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Edited by G. P. Youvaraj

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3. general survey articles, popular articles, expository papers and Book-Reviews.
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RELEVANCE OF MATHEMATICAL SCIENCES IN NATIONAL DEVELOPMENT, ITS CURRENT STATUS, MATHEMATICS PHOBIA AND ITS PREVENTION STRATEGIES

S. S. KHARE

Abstract. The main objectives of the Presidential Address is (i) to highlight the role and importance of mathematics in National development, intellectual development and in enhancing disciplinary and moral values among students, (ii) to describe its current status, and reasons of prevailing mathematics phobia among school students and (iii) to make some suggestions to help reducing the fear for mathematics to a great extent.

1. Introduction

The Indian Mathematical Society was founded on the 4th of April, 1907 by V. Ramaswami Aiyar with 20 founding members under the name "Indian Mathematics Club" having headquarter at Pune and B. Hanumant Rao as its first President from 1907 to 1912. The Society acquired its present name the "Indian Mathematical Society” in 1910, when it’s constitution was adopted. There have been 78 Presidents of the Society till now. The first conference of the Society was held at Madras in 1916. Till 1951, conference used to be held every 2 years or 3 years. But in the 1951 Conference, it was decided to hold conference of the society every year. The Society started publishing research journal with the name "The Journal of Indian Mathematical Society" from 1910 with its first Editor M.T. Naraniengar. Celebrated renowned mathematician Srinivasa Ramanujan published a short paper consisting of some questions and another paper of

This article is based on the Presidential talk (general) given by Prof. S. S. Khare, the president of the IMS, in the 91st Annual Conference of the IMS-An International Meet held at Lucknow Unversity, Lucknow during December 26-29, 2025.

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15 pages on "Some properties of Bernoulli Numbers" in The JIMS in 1911. He published his 12 papers in this Journal.

The Society decided in 1932 to publish another journal with the name "The Mathematics Student" with its first Editor A. Narasinga Rao. The J. of IMS aims at publishing high quality research papers, while The Mathematics Student aims at publishing Presidential Address, Planary talks, Award lectures given in the Annual Conference, survey articles, popular articles, expository articles, book reviews, problems and solutions, clever new proofs of known theorems etc.

The main objectives of the Society are to promote the quality research in mathematics, to inspire and encourage researchers, mathematics educators and students, and to popularize mathematics in India. The Society has been achieving these objectives by organizing Annual Conference, lectures, Symposiums, Award lectures, Plenary lectures, IMS sponsored lectures and by publishing The J. of IMS and The Mathematics Student. I feel that IMS has yet to do a lot for popularizing mathematics among young students at High-School and +2 level which is the foundational stage for graduate level. Mathematics is considered as one of the greatest and beautiful creation of human race and it has historically been India's strength from Acharya Pingla, Aryabhat I, Bhaskara I, Varahamihira, Bramhagupta, Mahaviracharya, Aryabhat II, Sridharacharya, Bhaskaracharya II, Madhava, Neelkanth, Ramanujan, Mahalanobis, C.R. Rao, Harish Chandra, Manjul Bhargav and Akshay Venkatesh etc.

2. Role of Mathematical Sciences in National development

In 21st century, mathematical sciences are not just for academic pursuit, but a strategy enabler of economic growth, technological innovation, governance and national security etc. As India moves towards becoming a 10 trillion dollar economy and a global knowledge hub by 2035, the role of mathematical Sciences becomes more vital and central. Key roles of mathematics for the development of our country are as follows:

(1) **Core of STEM:** Mathematics is the backbone of science, technology and engineering enabling technological and scientific breakthroughs.
(2) **Increasing productivity:** Strong scientific and technological foundation powered by mathematics, lead to increased economic productivity and competitiveness for our country.

(3) **Technological solutions:** Mathematical concepts, methods and tools are indispensable for the functioning of high-tech society allowing for the development of new and new technologies and more efficient systems.
(4) **Economic planning:** Mathematical models and statistics are heavily used for economic forecasting, risk management, investment strategies and financial management providing crucial data and its analysis for economic planning.
(5) **Engineering and design:** Modern designs and complex structures, especially in high-tech fields, require good mathematical understanding and numerical data, ensuring robust and efficient infrastructure.
(6) **Sector specific progress:** Mathematics supports development across all sectors including defence, education, healthcare and manufacturing by providing tools for data analysis, prediction and problem solving.
(7) **Artificial Intelligence and Machine learning:** During the last 2 decades, the need of machine learning and artificial intelligence has grown dramatically. A critical component of this is the ability to analyse very large, high dimensional complex and unstructured data. Earlier, graphs and statistical tools etc. were used for data analysis. But they are not successful in unstructured high-dimensional large data. These days, tools of Algebraic Topology such as persistent homology, persistent combinatorial Laplacians are used for analysis of such data.
(8) **Improving decision making:** A good grasp of mathematical concepts helps in improving in decision making in governance, business and in daily life leading to more effective and systematic approaches to complex challenges.

Realizing the potential of mathematical sciences in the rapid development, the Society of Industrial and Applied Mathematics was established in 1952 in USA. The Society contributed significantly towards development of Industrial Mathematics, which in turn helped in scientific, technological and industrial development of USA. Later, International Congress of Industrial and Applied Mathematics (ICIAM) was established with its first meeting in Paris in 1987 with main objective to promote those branches of mathematics, which are more relevant to economic, scientific, technological and industrial development. It's 10th conference was held in Tokyo in 2023. In our country also, Indian Society of Industrial and Applied Mathematics (ISIAM) was established in 1990 to promote the teaching and research in mathematics with applications in science, engineering, medical and social sciences and industries. ISIAM was affiliated to ICIAM during the 5th meeting of ICIAM in Edinberg in 2003. As per the recommendation of ICIAM and ISIAM, following 18 important branches of mathematics need to be focussed to meet the objectives. They are computational mathematics, optimization, image processing, machine learning, artificial intelligence, financial mathematics, industrial mathematics, data analysis, nano- mathematics, computer simulation, bio-mathematics, molecular modelling, prediction analysis, wavelets and their applications, fuzzy mathematics and applications, modelling and simulation for industries, application of mathematics to agriculture, environmental analysis. ISIAM publishes "The Indian J. of Industrial and Applied Mathematics" to promote research in the field. NBHM sponsors mathematician to attend ICIAM and ISIAM conferences. ISIAM also publishes material relevant to industrial, economic and technological growth.

3. Role of mathematics in intellectual development and in enhancing disciplinary and moral values

In mathematics, mostly we are given some hypothesis, a set of some known relevant results and from this one is required to prove some new result deductively or by method of contradiction. Analysing the given hypothesis, one proceeds step by step logically using prescribed rules of mathematics and at times using cleverly some earlier known results judicially, one concludes the final result to be proved.

The habit of carefully analysing the hypothesis, the habit of logically and truthfully applying prescribed rules of mathematics, the habit of proceeding step by step logically and the habit of judicially choosing and applying specific earlier known results to prove the final result help in magnetizing the brain of a serious mathematics practitioner in the sense that analytical, rational, critical, logical, imaginative and innovative skills of the brain get enhanced. Thus a decent mathematics training helps significantly in intellectual development and acts as sharpener for brain.

How mathematical training helps in enhancing organized and systematic thought, may be illustrated by an example in physics about the process of magnetizing an ordinary iron bar. The only difference between an ordinary iron bar and a magnetic iron bar is that the iron molecules in ordinary iron bar are spread in unorganized and haphazard manner, while the same iron molecules in magnetic iron bar are arranged in organized manner as shown in the Figure 1 and Figure 2.

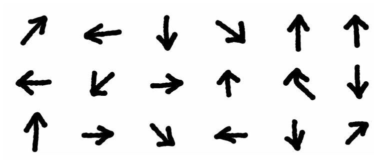

FIGURE 1. Iron molecules spread in haphazard manner in the ordinary iron bar

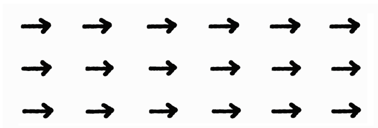

FIGURE 2. Iron molecules arranged in organized manner in the magnetic iron bar

When a magnet is rubbed "properly" on an ordinary iron bar, it also gets magnetized. In this process, the unorganized iron molecules of the ordinary iron bar get organized. Similarly, after practicing mathematics "properly", the unorganized thoughts of brain get organized, making it more sharp and powerful.

Apart from mathematics playing a key role in national development, and mathematics with computer proficiency having a tremendous job potential in various sectors, I feel that the biggest hidden gift of mathematics is that a proper mathematical training helps significantly in making brain more powerful, sharp and imaginative, which is a key factor for being successful in any walk of life, be it administration, be it medical profession, be it engineering, be it scientist, be it teaching , be it business, be it management, and be it even politics. Even enhancement of such skills of brain may help in being a good father/mother, a good brother/sister, a good husband/ wife, a good son/daughter and good friend as well.

Also, while practicing mathematics, one develops habit of truthfully following the rules of mathematics which helps the person to be more truthful, law abiding and disciplined in his/her day to day life. As mathematics deals with accurate and precise facts and arguments with no scope of vagueness, it helps a mathematics practitioner to express his/her views on any topic more precisely, to the point and without any vagueness. Mathematics is universal and is not dependent on cast, creed, religion, country, region etc. This makes a mathematics practitioner more open, broad minded and unbiased. He/she may develop habit of universal acceptance without any bias of cast, language and religion.

4. Current status of Mathematics

Despite the fact that mathematics has such a wholesome and universal role for development, the status of mathematics in our country is not satisfactory, especially at High-school and +2 level. There is a general perception among school students that mathematics is a dry, boring and hard nut to crack. This has caused a phobia for mathematics among most of the students. Some of the key reasons for maths phobia are as follows:

(1) Over emphasis on rote learning and memorizing, instead of understanding.
(2) Inadequate use of visual, practical or activity based learning.
(3) Lack of real life connection with mathematical concepts, while teaching.
(4) Lack of participatory teaching. Students are discouraged to raise questions in the class.
(5) Absence of mathematics lab in schools.
(6) Lack of use of teaching aids like videos on mathematics, softwares (GeoGebra, Desmos), Smart whiteboard and AI powered Tutors etc.
(7) Lack of books and written material focussing on understanding of concepts, on connection of mathematical concepts with real-life situation, and on simple concrete applications.
(8) Some what heavy syllabus and pressure on teachers to complete syllabus in time, giving good result. This also causes stereotype and

mechanical teaching without focusing on deep thinking, understanding and and creativity.

(9) Marking of questions on the basis of final answer, not on procedure is a common practice. Even if procedure is correct and due to some unforeseen mistake in addition or subtraction, if answer is wrong, mostly zero mark is given. Such harsh and irrational marking causes frustration, turning to fear with mathematics.
(10) Multiple choice or True/False questions with very little time create undue pressure on students to get right answer quickly. This discourages deep and cool thinking. Students start believing that mathematics is for quick thinkers and intelligent ones.
(11) Considering marks as sole criteria for intelligence and success by teachers and parents forces students to get more marks by hook or crook without focussing on understanding.
(12) Examination system mostly tests memorization potential, no reasoning, understanding and creativity.

5. Some Suggestions

In order to reduce mathematics phobia significantly and in order to make mathematics "student friendly", I would like to give some suggestions:

(1) The focus of mathematics teaching needs to be shifted to mathematizing the thought process of students i.e. to develop clarity of thought process to enhance systematic, analytical, critical, rigorous, logical and creative skills of brain.
(2) Effort should be made to use charts, models, demonstrations, hand-on activities, softwares like GeoGebra, Desmos, Nearpod etc, smart white board and mathematics videos etc to convey mathematics in interesting manner.
(3) Effort should be made to relate mathematical concepts with real-life situations to help students to get convinced that mathematics is not just an abstract subject, but also has relevance to our real lives.
(4) Mode of assessment in tests and examinations should be changed to test the understanding of the subject matter instead of cramming ability.
(5) Participatory teaching needs to be encouraged. Students should be encouraged to ask questions.

(6) Frequent training programmes for teachers need to be organized to update them with new text, procedures, techniques, pedagogy and also update them with the use of software, smart whiteboard etc. This will help teachers to convey mathematics in student-friendly manner.

(7) Writing of good books and teaching-learning material giving emphasis on understanding, explaining mathematics underlying various procedures along with brief historical notes and some simple concrete applications, and connection of different concepts with real-life situation is the need of hour.

(8) Inserting some humours, healthy relevant jokes, puzzles, paradoxes, fallacies, some recreational aspects of mathematics in between lectures help in making the class lively. Similarly, narrating some short stories or interesting incidents about great mathematicians and scientists or interesting stories about some mathematical/scientific discoveries in between serious mathematics may help in keeping the attention of students intact on one hand and in letting them know about great mathematicians and discoveries on the other hand, which may inspire some students.

(9) Students should be encouraged to solve some difficult problems in group. Collaborative project work should also be encouraged.

(10) Mathematics lab should be established in schools, which may help in understanding mathematics better through hand-on activity.

(11) In order to train the school teachers, we need to have a National Mathematics Master Training Programme to ensure creating a set of Master trainers in each district.

(12) We may create Mathematical Mentoring Circles i.e. networks of experience mathematics educators who may support school teachers online for best teaching practices and resources etc.

(13) It may be good idea to introduce online school teachers communities for sharing their practices and problems.

(14) Somewhat overloaded syllabus in mathematics may be reduced to ensure more time to focus on core competency, logical and analytical reasoning.

(15) One or two lines short justification should be made mandatory for multiple choice or True/False type questions to ensure whether student has ticked the answer by guess or by proper reasoning.

(16) The university and college teachers may be requested to interact with students and school teachers through periodic talks or discussion.

(17) The world of mathematics is full of wonders and amazing stuff, easily understandable. It may be good idea to supplement classroom teaching with fortnightly popular talks on such stuff giving inside beauty and elegance of mathematics.

(18) School libraries should buy some such books on recreational and popular mathematics and students may be encouraged to read such books in their leisure time. This will certainly help create interest in mathematics.

(19) For Various activities and measures mentioned above to be taken in order to make mathematics more palatable, teachers may need more time. One additional class per week for mathematics may fulfil the requirement of additional time.

(20) Through understanding oriented participatory teaching, using various teaching aids, giving real-life examples and applications, students will start finding interest in mathematics and will be convinced that although it may be somewhat hard nut to crack but once it is cracked, one will find inner beauty, elegance and wonders. In fact mathematics is like hairy, shabby looking hard coconut from outside and is quite hard to crack, but once it is cracked, one gets sweet and white coconut meat and sweet nutritions coconut juice.

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APPLICATION OF ALGEBRAIC TOPOLOGY IN DATA ANALYSIS AND LATEST TRENDS

S. S. KHARE

ABSTRACT. The objective of the Presidential address (Technical) is to introduce Topological Data Analysis (TDA) and to show the importance of of TDA in capturing inherent shape and structure of large size, complex, high dimensional, and even unstructured data, specially in continuous phenomenon. For this, we introduced point cloud data and clustering, loops and fundamental groups, simplicial complex, simplicial homology, persistent chain complex and persistent homology, and its applications to DNA structure, data analysis and stock exchange data analysis in detail. We have also given brief account of applications to cellular and gene data analysis, network analysis, computer vision data analysis, market dynamics data analysis and brain activity data analysis. At last, we have given applications of other tools of Algebraic Topology in data analysis such as sheaves and cosheaves, discrete Morse Theory, homotopy groups, cohomology and cup products, spectral sequence, persistent combinatorial Laplacian, and persistent Dirac operator etc.

1. INTRODUCTION

In the last two to three decades, the need of artificial intelligence and machine learning has grown dramatically. As the tasks undertaken become more ambitious, the data often becomes very large, complex and high-dimensional, giving challenge to develop methods good enough to analyse such data. The traditional methods were best suited for structured limited source data, but were not able to capture the inherent shape and structure

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This article is based on the Presidential talk (Technical) given by Prof. S. S. Khare, the president of the IMS, in the 91st Annual Conference of the IMS-An International Meet held at Lucknow Unversity, Lucknow during December 26-29, 2026.
2020 Mathematics Subject Classification: 55N31, 35, 55U10, 62R40, 68T09
Key words and phrases: Algebraic Topology, Topological Data Analysis, persistent homology, simplicial complex

of complex, high-dimensional and large size data, especially for continuous phenomenon.

In order to meet this challenge, Topological Data Analysis (TDA) was introduced around 2000 with the foundational work on persistent homology by Edelsbrunner et al, and Zomorodian and Carlsson. TDA could address the challenge of analysing such complex and even unstructured data for continuous phenomenon successfully by applying tools of Algebraic Topology. Persistent homology, a key tool of Algebraic Topology, has been found suitable in analysing multi-scale topological features like connected components, holes (one dimensional loops) and voids (high-dimensional holes). Persistent homology is also able to capture those topological features of the data which persist with the change of scale or change of time span.

2. POINT CLOUD DATA AND CLUSTERING

A point cloud data is a random collection of data points normally in $\mathbb{R}^3$ represented through coordinates. By clustering of data, we mean partitioning the data into a number of parts or clusters, points in the same cluster being similar in some sense and points in different clusters being not similar. In the context of finite metric space, this means roughly that points within a cluster are nearer to each other than they are to points in other clusters.

Points in different clusters
Fig. (1)

During some collections, some parts of the data may have been taken under tighter threshold, while some parts of the data may have been taken under looser threshold. In such situation, one prefers to divide the complete data X into two (or more subsets), say X_1, and X_2 with $X = X_1 \sqcup X_2$ such that the division is consistent in the following sense.

Let clusters for X, X_1 and X_2 be $C(X), C(X_1)$ and $C(X_2)$ respectively. The division of X into X_1 and X_2 is said to be consistent if the clusterings $C(X_1)$ and $C(X_2)$ on X_1 and X_2 correspond to the clustering $C(X)$ of X as shown in the Fig. (2).

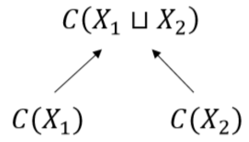


Fig. 2

3. LOOPS, FUNDAMENTAL GROUP, AND HIGHER HOMOTOPY GROUP

Consider two spaces on a surface given by the following images

Both spaces are connected and path connected. These topological properties are not able to distinguish the two spaces. A loop in a topological space X at a point x_0 in X is a continuous map $f : S^1 \to X$ with $f(s_0) = x_0, s_0$ being a base point in circle S^1. Note that the space A has only one loop, while the space B has two loops. The space A can be continuously deformed to a circle O and the space B can be continuously deformed to figure ∞, which is one point union of two circles. Note that the figure ∞ can not be continuously deformed to the figure O.

Given a topological space X and $x_0 \in X$, let $L(X; x_0)$ be the set of all loops in X at x_0. The relation "continuously deformic to" is an equivalence relation in $L(X; x_0)$. The set $\pi_1(X; x_0)$ of equivalence classes forms a group with respect to operation

$$[\alpha] \circ [\beta] = [\alpha * \beta],$$

where $[\alpha]$ is the equivalence class of the loop α and the join $\alpha * \beta$ of α and β is defined as $\alpha * \beta : [0, 1] \to X$ given by

$$\alpha * \beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}, \\ \beta(2t - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The group $\pi_1(X; x_0)$ is called the fundamental group of X at x_0. The number of generators in $\pi_1(X; x_0)$ for path connected space X gives the number of holes in X. Note that the fundamental group of the circle is the group

Z of integers having one generator. Also, $\pi_1(\infty) \approx Z \oplus Z$ having two generators.

In the same manner, higher homotopy groups $\pi_n(X; x_0), n > 1$ can also be defined. An $n-$dimensional loop in X at x_0 is a continuous map $f : S^n \to X$ with $f(s_0) = x_0, s_0$ being a fixed base point in S^n. The relation "continuously deformic to" is an equivalence relation in the set $L_n(X; x_0)$ of all $n-$dimensional loops in X at x_0 and the set $\pi_n(X; x_0)$ of all equivalence classes form a group with suitable operation. The group $\pi_n(X; x_0)$ is called the $n-$dimensional homotopy group of X at x_0.

Unlike π_1, the number of generators in $\pi_n(X; x_0)$ does not represent number of $n-$dimensional holes. For example $\pi_3(S^2) = Z$ has one generator, but it does not correspond to single $3-$dimensional hole in S^2. There is no $3-$dimensional hole in S^2.

4. Simplicial Complex

Let $\{a_0, a_1, \ldots, a_n\}$ in $\mathbb{R}^n$ be geometrically independent, i.e., vectors $(a_1 - a_0), (a_2 - a_0), \ldots, (a_n - a_0)$ being linearly independent. The n-simplex σ spanned by $\{a_0, a_1, \ldots, a_n\}$ is the set of all points

$$\sigma = \left\{ \sum_{i=0}^{n} t_i a_i : \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \right\}.$$

The points $a_0, a_1, \ldots, a_n$ are called the vertices of the simplex σ of dimension n. The simplex spanned by a proper subset of $\{a_0, a_1, \ldots, a_n\}$ is called a face of σ. Let $e_i \in \mathbb{R}^{n+1}$ be $\{0, 0, \ldots, 1, \ldots, 0\}$ with 1 at the ith place and 0 elsewhere. The n-simplex spanned by $\{e_0, e_1, \ldots, e_n\}$ is called the standard n-simplex and is denoted by Δ^n.

Definition 4.1. A simplicial complex K in $\mathbb{R}^N$ is a collection of simplices σ in $\mathbb{R}^N$ such that

(a) Every face of a simplex σ in K also belongs to K.
(b) The intersection of two distinct simplices in K is a face of both the simplices.

Example 4.2. The figure given below is a simplicial complex given by $\{\Delta ABC, \Delta ACD, AB, BC, AC, DC, AD, \{A\}, \{B\}, \{C\}, \{D\}\}$ with vertices A, B, C and D.

Example 4.3. The figure given below is not a simplicial complex, since the simplices ΔABC and ΔEFG do not intersect on their faces.

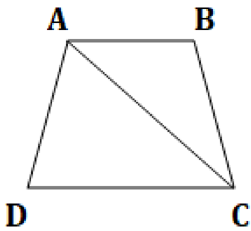


Fig.(3)

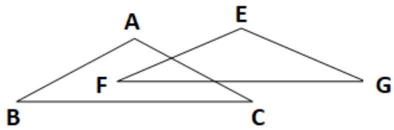


Fig.(4)

Definition 4.4. An abstract simplicial complex is a set $\mathcal{S}$ of finitely many non-empty sets as elements such that if $A \in \mathcal{S}$, then every nonempty subset of A also belongs to $\mathcal{S}$.

Every element A in $\mathcal{S}$ is called a simplex of dimension $|A| - 1$, where $|A|$ is the number of elements in A. A point a is said to be a vertex of the abstract simplicial complex if $\{a\} \in \mathcal{S}$.
Given an abstract simplicial complex $\mathcal{S}$, one can assign a unique geometric simplicial complex K by assigning to each subset $A \in \mathcal{S}$, the simplex spanned by elements of A and vice-versa.

Example 4.5. Given an abstract simplicial complex $\{\{A, B, C\}, \{A, C, D\}, \{A, B\}, \{B, C\}, \{A, C\}, \{D, C\}, \{A, D\}, \{A\}, \{B\}, \{C\}, \{D\}\}$, one can assign a unique geometric simplicial complex given by

$$\{\Delta ABC, \Delta ACD, AB, BC, AC, DC, AD, A, B, C, D\}.$$

Converse is also true.

Definition 4.6. A triangulation of a topological space X is a homeomorphism $f : K \to X$, where K is a geometric simplicial complex. In this case, X is said to be triangulable.

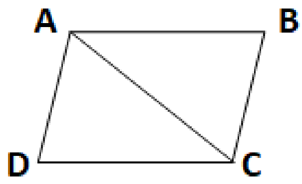
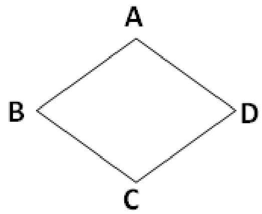
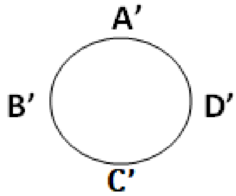


Fig.(5)

Example 4.7. A triangulation of a circle S^1 is a homeomorphism $f : K \to S^1$, where

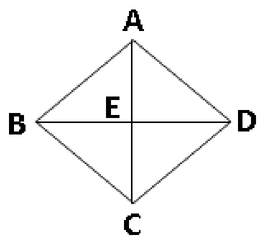
$K =$ 

$$= \{AB, BC, CD, DA, A, B, C, D\}$$

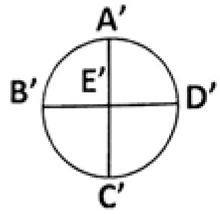
and $f : K \to S^1 =$ 

given by $f(A) = A', f(B) = B', f(C) = C', f(D) = D', f(AB) = \text{arc } A'B'$, $f(BC) = \text{arc } B'C', f(CD) = \text{arc } C'D', f(DA) = \text{arc } D'A'$.

Example 4.8. A triangulation of the disc D is a homeomorphism $f : K \to D$, where the simplicial complex K is given by



$= \{\Delta ABE, \Delta BCE, \Delta CDE, \Delta DAE, AB, BC, CD, DA, AE, BE, CE, DE,$

$A, B, C, D, E\}$ and $f : K \to D \; =$ 

given by

$$f(\Delta ABE) = \text{Quadrant } A'B'E', f(\Delta BCE) = \text{Quadrant } B'C'E',$$

$$f(\Delta CDE) = \text{Quadrant } C'D'E', f(\Delta DAE) = \text{Quadrant } D'A'E',$$

$$f(AB) = \text{arc } A'B', \quad f(BC) = \text{arc } B'C', \quad f(CD) = \text{arc } C'D',$$

$$f(DA) = \text{arc } D'A', \quad f(AE) = A'E', \quad f(BE) = B'E',$$

$$f(CE) = C'E', \quad f(DE) = D'E'.$$

5. Simplicial Complexes from a Point Cloud Data

Given a point cloud data $X = \{x_1, x_2, \ldots, x_n\}$, consider X_ϵ as the union of all closed balls $\overline{B_\epsilon(x_i)}$ with center x_i and radius ϵ. Thus

$$X_\epsilon = \bigcup_{i=1}^{n} \overline{B_\epsilon(x_i)}.$$

X_ϵ is called the ϵ-offset or ϵ-thickening of X. Clearly, $\{\overline{B_\epsilon(x_i)}\}_i$ is a covering of X.

Definition 5.1. Given a cover $\mathcal{U} = \{u_\alpha\}$ of a topological space, the nerve $N(\mathcal{U})$ of the cover $\mathcal{U}$ is the abstract simplicial complex whose k-simplices are determined by $(k+1)$ elements of $\mathcal{U}$ having nonempty intersection. Thus

$$[u_{i_0}, \ldots, u_{i_k}] \in N(\mathcal{U}) \Longleftrightarrow \bigcap_{n=0}^{k} u_{i_n} \neq \phi.$$

Theorem 5.2 (Nerve Theorem). *Given a cover $\mathcal{U} = \{u_\alpha\}_{\alpha \in A}$ of a space X such that for any $A' \subset A$,*

$$\bigcap_{\alpha \in A'} u_\alpha$$

is either ϕ or contractible, then X is homotopically equivalent to the nerve $N(\mathcal{U})$ of $\mathcal{U}$.

Definition 5.3. The Čech complex $\check{C}_\epsilon(X)$ of a point cloud $X = \{x_1, \ldots, x_n\}$ for a given ϵ is defined as the abstract simplicial complex given by the nerve of the covering $\{\overline{B_\epsilon(x_i)}\}_{i=1}^{n} = X_\epsilon$ of X.

Definition 5.4. The Rips complex $R_\epsilon(X)$ of a point cloud $X = \{x_0, x_1, \ldots, x_n\}$ for a given ϵ is defined as the abstract simplicial complex whose k-simplices are determined by $(k+1)$ points $\{x_{i_1}, \ldots, x_{i_{k+1}}\}$ from the point cloud X such that $\{x_{i_1}, \ldots, x_{i_{k+1}}\}$ are pairwise less than or equal to ϵ distance apart, i.e.

$$d(x_{i_r}, x_{i_s}) \leq \epsilon, \quad 1 \leq r, s \leq k+1.$$

Example 5.5. Consider a point cloud X with points $x_1, x_2, \ldots, x_7, x_8$.

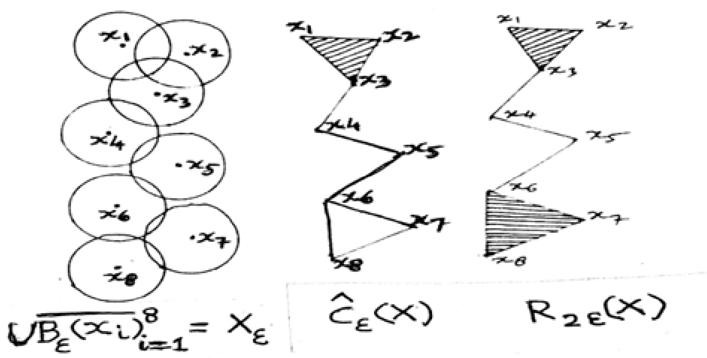


Fig 6

Remark 5.6. $\hat{C}_\epsilon(X) \subset R_{2\epsilon}(X) \subset \hat{C}_{2\epsilon}(X)$.

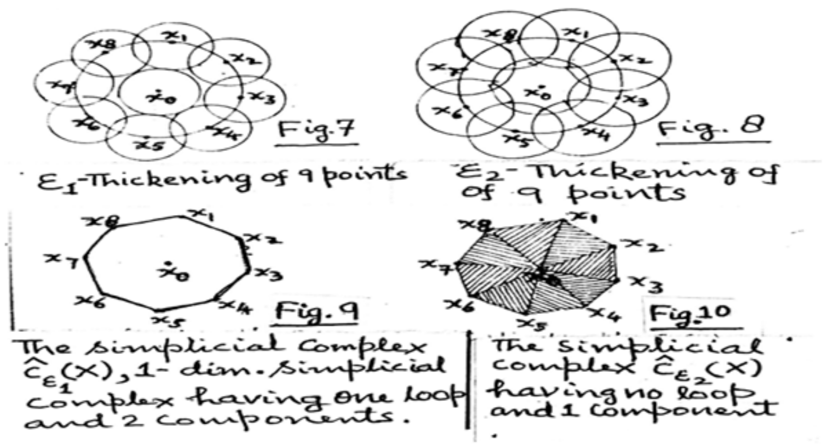
Remark 5.7. The number r of intersections that need to be computed to determine the Čech complex of a large size point cloud X often poses some practical problems. On the other hand, the ϵ-connectivity graph is better tractable to compute, making the Rips complex $R_\epsilon(X)$ a more computationally viable option.

Remark 5.8. Čech complexes (or Rips complexes) for different ϵ_1 and ϵ_2 may have totally different features.

Example 5.9. Consider a point cloud $X = \{x_0, x_1, \ldots, x_7, x_8\}$ and $\epsilon_1 < \epsilon_2$. The sketches of ϵ_1 and ϵ_2 thickening of 9 points and of simplicial complexes $\hat{C}_{\epsilon_1}(X)$and$\hat{C}_{\epsilon_2}(X)$ are as follows:

6. Simplicial Homology

Given a simplicial complex K, the free abelian group generated by all the p-dimensional simplexes in K is called the group of all p-chains in K and is



denoted by $C_p(K)$. The pth boundary map $\partial_p : C_p(K) \to C_{p-1}(K)$ is the map given by

$$\partial_p([v_0, v_1, \ldots, v_p]) = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \ldots, \widetilde{v_i}, \ldots, v_p],$$

where $[v_0, v_1, \ldots, v_p]$ is the p-simplex spanned by vertices $v_0, v_1, \ldots, v_p$ and $[v_0, v_1, \ldots, \widetilde{v_i}, \ldots, v_p]$ is the $(p-1)$-simplex spanned by the p vertices $v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p$. One has the following chain complex for any simplicial complex K

$$\cdots \to C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} \cdots$$

with $\partial_p \partial_{p+1} = 0$, so that $\operatorname{Im} \partial_{p+1} \subset \ker \partial_p$.
Elements of $\ker \partial_p$ are called p-cycles and elements of $\operatorname{Im} \partial_{p+1}$ are called p-boundaries. The quotient group $\ker \partial_p / \operatorname{Im} \partial_{p+1}$ is called the pth homology group of the simplicial complex K and is denoted by $H_p(K)$. Note that $H_1(\check{C}_{\epsilon_1}(X)) = \mathbb{Z}$, since $\check{C}_{\epsilon_1}(X)$ given in Fig. (9) is continuously deformic to a circle S^1. Also, $H_1(\check{C}_{\epsilon_2}(X)) = 0$, since $\check{C}_{\epsilon_2}(X)$ given in Fig. (10) is continuously deformic to a point.

7. PERSISTENT CHAIN COMPLEX AND PERSISTENT HOMOLOGY

A chain complex $\{A_i, d_i\}_i$ is a sequence of abelian groups or modules A_i, $i \in \mathbb{N} \cup \{0\}$ connected by boundary homomorphism $d_n : A_n \to A_{n-1}$ satisfying $d_n d_{n+1} = 0$.

Example 7.1. As mentioned in section (6), given a simplicial complex K,

$$\cdots \to C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \xrightarrow{\partial_{p-1}} \cdots$$

is a chain complex.

Definition 7.2. A persistent complex $\mathcal{C}$ consists of a family $\{\mathbf{C}_i\}$ of chain complexes, $i \in \mathbb{N} \cup \{0\}$ along with R-module homomorphisms

$$\mathbf{f}_n^i : \mathbf{C}_n^i \longrightarrow \mathbf{C}_n^{i+1}, \qquad i, n \in \mathbb{N} \cup \{0\}$$

making the following diagram commutative.

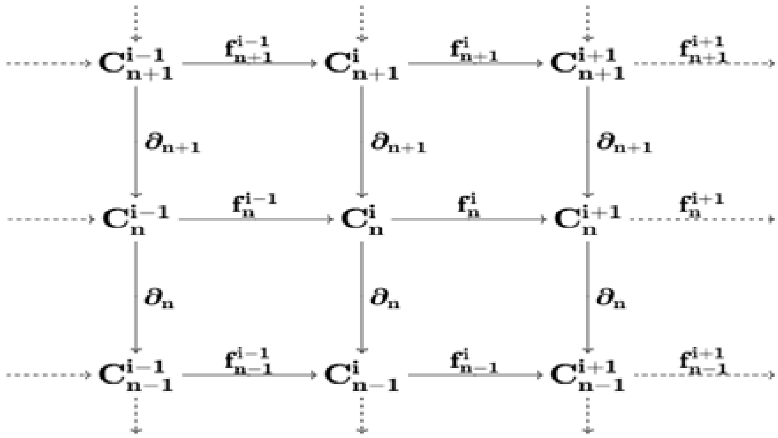


Fig. (11)

Example 7.3. Given a simplicial complex K, let K_i be the simplicial complex consisting of all simplexes of dim $\leq i$. Consider the filtration

$$K_0 \hookrightarrow K_1 \hookrightarrow \cdots \hookrightarrow K_{i-1} \hookrightarrow K_i \hookrightarrow K_{i+1} \hookrightarrow \cdots \hookrightarrow K$$

This gives a persistent complex at homology level as follows, Here $\mathcal{C} = \{\mathbf{H}_n(K)^i\}$.

Definition 7.4. A persistent complex $\mathcal{C}$ is said to be of finite type, if each R-module C_n^i is finitely generated and $\exists\ n$ such that for $i \geq n$, $f_i : C_i \to C_{i+1}$ is an isomorphism of chain complexes.

Remark 7.5. If the simplicial complex K is finite, then the persistent complex given in Example (9) will be of finite type, since each homology module $H_n(K_i)$ will be of finite type.

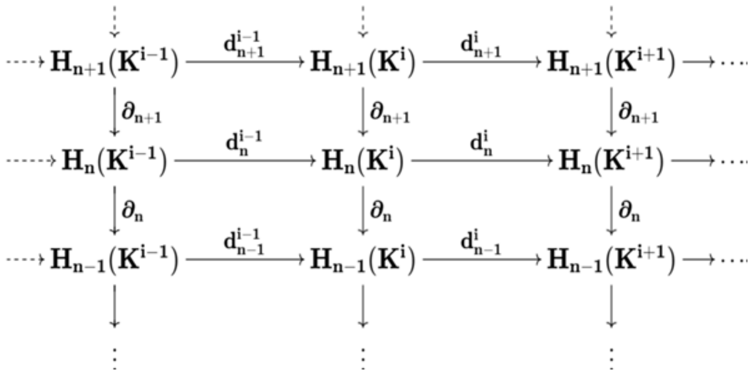


Fig. (12)

Example 7.6. For a point cloud X and given an increasing sequence of real numbers $\{\epsilon_i\}, i = 0, \ldots, n$, consider the family $\{\hat{C}_{\epsilon_i}(X)\}$ of $\hat{C}$ech complexes. Consider the filtration of $\hat{C}_{\epsilon_i}(X)$

$$\hat{C}^0_{\epsilon_i}(X) \hookrightarrow \hat{C}^1_{\epsilon_i}(X) \hookrightarrow \cdots \hat{C}^j_{\epsilon_i}(X) \hookrightarrow \hat{C}^{j+1}_{\epsilon_i}(X) \hookrightarrow \cdots$$

where $\hat{C}_{\epsilon_i} j(X)$ is the simplicial complex consisting of all simplexes in $\hat{C}_{\epsilon_i}(X)$ of dimension $\leq j$. This filtration gives the following persistent complex at homology level.

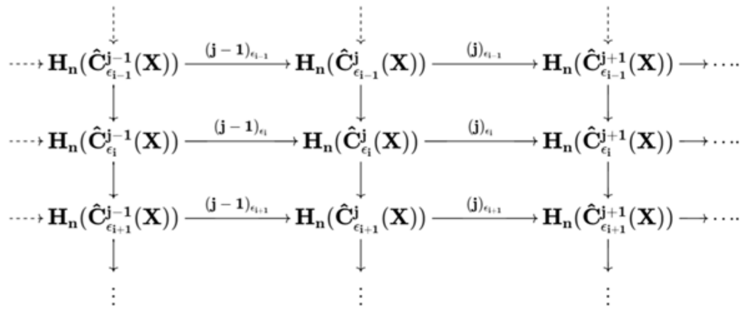


Fig. (13)

Remark 7.7. If the number of elements in the point cloud X is finite, then the persistent complex given above is also of finite type.

Definition 7.8. Given a simplicial complex K of dimension n and the filtration

$$K^0 \hookrightarrow K^1 \hookrightarrow \cdots \hookrightarrow K^i \hookrightarrow K^{i+1} \hookrightarrow \cdots \hookrightarrow K^n = K, \qquad 0 \leq i \leq n,$$

the inclusion map $K^i \hookrightarrow K^j$, $i < j$ induces a homomorphism

$$f_p^{i,j} : H_p(K^i) \longrightarrow H_p(K^j)$$

for every p, i and j. The p^{th} persistent homology groups $PH_p^{i,j}(K)$ are defined as the Image($f_p^{i,j}$) and p^{th} persistent Betti numbers $\beta_p^{i,j}(K)$ defined as the ranks of the groups Image($f_p^{i,j}$).

Definition 7.9. A persistent barcode represents each persistent generator of the persistent groups $PH_p^{i,j}(K)$ with a horizontal line segment beginning at the first filtration level where it appears and ending at the filtration level where it disappears.

Definition 7.10. A persistent diagram plots a point for each generator of the persistent groups $PH_p^{i,j}(K)$ with its x-coordinate the birth time and its y-coordinate the death time. Equivalently, the same data can be represented by a diagram where each generator is represented by a line segment connecting the birth and death values plotted for separate p. This diagram is called barcode diagram for p.

Definition 7.11. The p^{th} Betti number of a simplicial complex K is the rank of the group $H_p(K)$.

Example 7.12. Consider a point cloud X consisting of points x_1, x_2, x_3, x_4 with $d(x_1, x_2) = d(x_3, x_4) = 1$ and $d(x_1, x_3) = d(x_2, x_4) = 2$. An increasing sequence of ϵ produces a filtration, i.e., a sequence of increasing simplicial complexes (Rips Complexes)

$$R_{\epsilon_1}(X) \subseteq R_{\epsilon_2}(X) \subseteq R_{\epsilon_3}(X) \subseteq \cdots$$

Persistent homology tracks homology classes along the filtration.

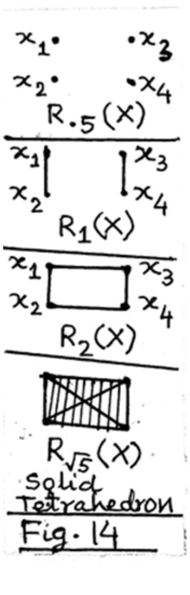


Fig. 14

At $\epsilon_1 = .5$, the Rips complex $R_{.5}(X)$ has 4 components with $H_0(R_{.5}(X)) \approx \mathbb{Z}^4$, showing that 0^{th} Betti number $\beta_0(R_{.5}(X)) = 4$. At $\epsilon_2 = 1$, the Rips complex has two edges and two connected components, i.e., $H_0(R_1(X)) \approx \mathbb{Z}^2$ so that the 0^{th} Betti number $\beta_0(R_1(X)) = 2$. At $\epsilon_3 = 2$, $R_2(X)$ is a 1-dimensional rectangle with one connected component, i.e., $H_0(R_2(X)) \approx \mathbb{Z}$, i.e., $\beta_0(R_2(X)) = 1$.

Also, $H_1(R_2(X)) \approx \mathbb{Z}$, showing that a one dimensional loop is born at $\epsilon_3 = 2$. There was no loop for $\epsilon < 2$. Also, at $\epsilon_4 = \sqrt{5}$, $R_{\sqrt{5}}(X)$ is a solid tetrahedron having no hole and continuously deformic to a point, showing that $H_0(R_{\sqrt{5}}(X)) \approx \mathbb{Z}$ and $H_1(R_{\sqrt{5}}(X)) = 0$. Thus $\beta_0(R_{\sqrt{5}}(X)) = 1$ and $\beta_1(R_{\sqrt{5}}(X)) = 0$. All this can be represented by persistent barcode as follows.

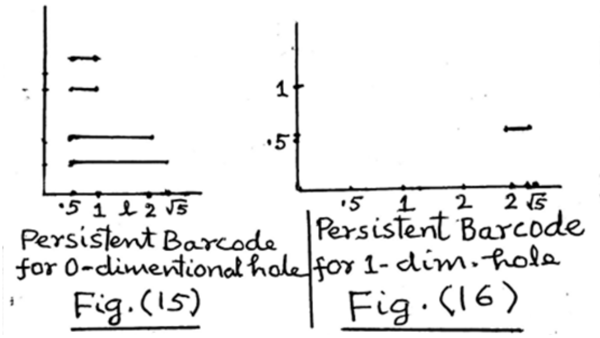
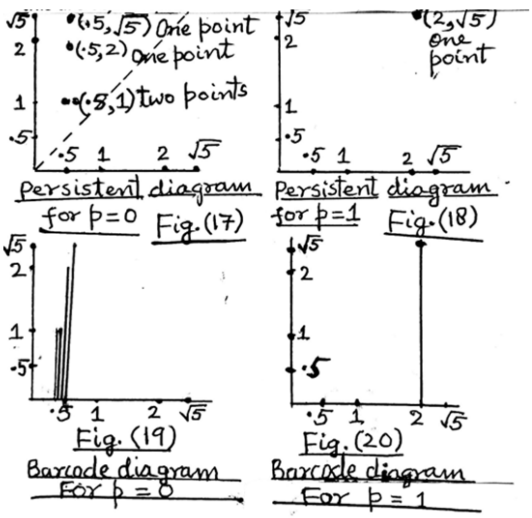


Fig. (15) shows 0-dimensional holes. At $\epsilon_1 = 0.5$, there are 4 bars for 4 connected components in $R_{.5}(X)$. At $\epsilon_2 = 1$, $R_1(X)$ has only two components. This is why the two top bars which started at $\epsilon_1 = 0.5$ die at $\epsilon_2 = 1$. At $\epsilon_3 = 2$, $R_2(X)$ is a one dimensional rectangle giving only one connected component with $H_0(R_2(\mathbb{Z})) = \mathbb{Z}$. Thus, the third bar from the top starting from $\epsilon_1 = .5$ also ends at $\epsilon_3 = 2$. But in $R_2(X)$, one 1-dimensional hole appears (inside rectangular edge). Note that $H_1(R_2(X)) \approx \mathbb{Z}$, showing that the Betti number $\beta_1(R_2(X)) = 1$. In fact number of generators of the homology group $H_1(R_2(X))$ reflects the number of 1-dimensional loops in $R_2(X)$. Also, since $R_{\sqrt{5}}(X)$ is a solid tetrahedron, $H_0(R_{\sqrt{5}}(X)) \approx \mathbb{Z}$ so that the bottom bar in Fig. (15) continues. Further, $H_1(R_{\sqrt{5}}(X)) = 0$. Thus, the one dimensional loop, which appeared at $\epsilon_3 = 2$, disappears at $\epsilon_4 = \sqrt{5}$. This fact is reflected in Fig. (16). The Persistent diagrams and Barcode diagrams for this are as follows.

For the point cloud $X = \{x_1, x_2, x_3, x_4\}$ and the increasing sequence of distance parameters $\epsilon_1 = .5, \epsilon_2 = 1, \epsilon_3 = 2, \epsilon_4 = \sqrt{5}$, consider the increasing sequence of simplicial complexes

$$R_{.5}(X) \hookrightarrow R_1(X) \hookrightarrow R_2(X) \hookrightarrow R_{\sqrt{5}}(X).$$



Applying homology functor for different p, one gets

$$\begin{array}{ccccccc}
H_0(R_{.5}(X)) & \longrightarrow & H_0(R_1(X)) & \longrightarrow & H_0(R_2(X)) & \longrightarrow & H_0(R_{\sqrt{5}}(X)) \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\
\mathbb{Z}^4 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}
\end{array}$$

The above sequence shows how components are disappearing. For $p = 1$, one gets the following sequence

$$\begin{array}{ccccccc}
H_1(R_{.5}(X)) & \longrightarrow & H_1(R_1(X)) & \longrightarrow & H_1(R_2(X)) & \longrightarrow & H_1(R_{\sqrt{5}}(X)) \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

The above sequence shows appearing and disappearing of 1-dimensional hole (loop).
Similarly, considering the filtration

$$R^0_{\sqrt{5}}(X) \to R^1_{\sqrt{5}}(X) \to R^2_{\sqrt{5}}(X) \to R^3_{\sqrt{5}}(X) = R_{\sqrt{5}}(X)$$

and applying homology functors for different p, one gets interesting informations.

$$\begin{array}{ccccccc}
H_0(R^0_{\sqrt{5}}(X)) & \longrightarrow & H_0(R^1_{\sqrt{5}}(X)) & \longrightarrow & H_0(R^2_{\sqrt{5}}(X)) & \longrightarrow & H_0(R^3_{\sqrt{5}}(X)) \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\
\mathbb{Z}^4 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}
\end{array}$$

This sequence shows disappearing of connected components. For $p = 1$, one gets

$$\begin{array}{ccccccc}
H_1(R^0_{\sqrt{5}}(X)) & \longrightarrow & H_1(R^1_{\sqrt{5}}(X)) & \longrightarrow & H_1(R^2_{\sqrt{5}}(X)) & \longrightarrow & H_1(R^3_{\sqrt{5}}(X)) \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\
0 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

The above sequence shows that three independent 1-dimensional loops with one common point appear at the one-dimensional skeleton $R^1_{\sqrt{5}}(X)$ of $R(X)$ and disappear at the 2-dimensional skeleton $R^2_{\sqrt{5}}(X)$ of $R_{\sqrt{5}}(X)$ [See Fig. (21)].

Fig.(21)

Further, for $p = 2$, one gets

$$\begin{array}{ccccccc}
H_2(R^0_{\sqrt{5}}(X)) & \longrightarrow & H_2(R^1_{\sqrt{5}}(X)) & \longrightarrow & H_2(R^2_{\sqrt{5}}(X)) & \longrightarrow & H_2(R^3_{\sqrt{5}}(X)) \\
\wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

The above sequence shows that one 2-dimensional hole appears at the 2-dimensional skeleton $R^2_{\sqrt{5}}(X)$ of $R_{\sqrt{5}}(X)$ and disappears at $R^3_{\sqrt{5}}(X)$.

Thus, using homology theory, we are able to find out which topological feature such as connected components or 1-dimensional loops or higher dimensional loops appear at what level of parameter ϵ or skeleton and disappear at what level.

8. APPLICATION OF PERSISTENT HOMOLOGY

Persistent homology has diverse applications in areas like biology, such as cancer research, neuroscience etc, computer science, such as computer vision, network analysis etc, artificial intelligence and finance, such as analysing financial market data etc. We illustrate couple of applications

(1)<u>DNA structure</u>

Persistant homology is used to analyze protein 3D-structure. Due to their complex folding patterns, protein are ideal candidates for topological analysis, as their shapes contain rich structural information that traditional geometrical or statistical tools might overlook.

Step 1: Extracting protein data as a point cloud:

Protein are made up of chains of amino acids, and their 3-D suture is often represented by tracing the positions of specific atoms (most commonly alpha carbon atoms C_α) along protein backbone. These (x_i, y_i, z_i) coordinates form a natural point cloud X, which can then be analysed using persistent homology.

Step 2: With the point cloud X in hand, we build filtration, i.e., a sequence of simplicial complexes generated by connecting points of the point cloud set X based on growing distance ϵ threshold. This process tracks how topological features like, connected components, 1-dimensional loops and higher order loops appear and disappear as the distance scale is changed. For computing filtration and persistent homology, many specialized softwares are used these days. Important among them are

(i) Ripser: It is a highly efficient fast C++/Python software primarly used in computational topology for analyzing the topological features of the data. One of its core functions is the computation of Rips complex and its persistent homology.

(ii) GUDHI Software Library: It is a C++ library with Python interface used for Topological Data Analysis. In particular, it helps in constructing simplicial complexes from a given data and in computing persistent hourology.

Step 3: The results of persistent homology computed in step 2 are typically visualized by persistent barcode and persistent diagram.

(a) Persistent barcode: Each topological feature is represented by a horizontal line segment indicating the stage of birth and death of the topological features and the length of the line segment indicating how long the features persist.

(b) Persistent diagram: Here, each topological feature is plotted as a point in a 2-dimensional plane with $x-$coordinate being birth stage and $y-$coordinate being the death stage. In persistent diagram, the points very close to the diagonal line represent features appearing and disappearing almost immediately. Such points usually represent noise or error. The points farther from the line $y = x$ represent that the concerned topological feature is meaningful topological feature revealing the true structural patterns within protein.

Step 4: Interpreting Betti numbers: Using Betti numbers, the topological features are quantified.

(a) β_0 represents the number of connected components.
(b) β_1 represents the number of 1-dimensional loops.
(c) β_2 represents the number of voids (2-dim. loops/holes).

Step 5: Computing homology generators: While Betti numbers tell us how many topological features exist, it is also useful to identify the stage where the topological features appear and the stage where features disappear. This requires computation of generators of homology groups. For this, mostly the following software libraries are used

(a) Eirene Software Library: It is designed for advanced persistent homology group computation.
(b) GUDHI

(2) Application of persistent homology to Stock Exchange data
We illustrate it by the following example.
Suppose we track daily closing prices of 3 companies A, B and C over 100 trading days. Each day's price of the three companies form a point

$$P_t = (Price_A(t), Price_B(t), Price_C(t)), 1 \leq t \leq 100.$$

These 100 points form a point cloud X in the 3-dimensional space $\mathbb{R}^3$, which may be plotted. There may be two cases.

(a) If points lie near the diagonal line, it shows that the companies are correlated.
(b) If clusters of points appear away from the diagonal line, it shows that the companies are not correlated and fluctuate differently.

Thus, the observation of the plotted points give some qualitative general informations. In order to get structured information from the data, we need persistent homology. For this, as described earlier, we associate $\hat{C}$ech simplicial complex or Rips simplicial complex to the point cloud X for different distance parameters ϵ. From the filtration of the simplicial complex, one calculates persistent homology which tracks different topological features that appear and disappear as ϵ changes.

If at $\epsilon_0, 0^{th}$ homology $H_0 \approx \mathbb{Z}$, then the simplicial complex is connected showing that all the three stocks are correlated to each others. In case, H_0 has more than one generator for ϵ_0, then the simplicial complex is not connected and it shows that some stock move in uncorrelated manner (like tech stock vs pharma stock).

If at ϵ_1, H_1 is non-zero, then the simplicial complex has 1-order loop and it indicates cyclic trading pattern of stocks
(e.g. tech $\longrightarrow$ banking $\longrightarrow$ energy $\longrightarrow$ tech). See Fig. (22).

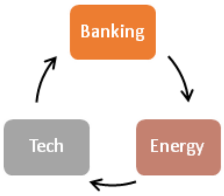


Fig.(22)

9. SOME MORE APPLICATIONS(BRIEF MENTION)

(A) Cellular and gene data analysis:
Persistent homology is also used to analyze complex data sets from primary visual cortex cell population and cancer gene expression data, providing insights into the underlying cellular organization.

(B) Network Analysis:
As a feature extractor, persistent homology summarizes complex networks, such as financial correlation networks or social networks etc, across multiple scales, providing a more comprehensive view.

(C) Computer vision:
Persistent homology is applied to extract topological features from computer images, and high dimensional point clouds, which helps in tasks such as scene understanding and object recognition.

(D) Market Dynamics:
Persistent homology helps in identifying patterns and anomalies in financial data, such as Bitcoin behaviour and stock market dynamics etc.

(E) Feature extraction:
Persistent homology helps in analyzing the topological information from a time series data, such as data of daily stock prices or data obtained from

hourly temperature reading etc. Such analysis helps in forecasting future values, detecting anomalies and understanding performance invariance fields like engineering, health monitoring and finance etc.

(F) Brain activity:

Persistent homology helps in detecting epileptic seizure, such as sudden burst of abnormal electrical signals, by identifying synchronized patterns in EEG signals (These signals record electrical activity of the brain).

10. SOME OTHER TOOLS OF ALGEBRAIC TOPOLOGY BEING USED FOR TDA IN RECENT TIME

We have seen use of simplicial complexes and persistent homology for TDA. However modern topological data analysis also uses some other tools of Algebraic Topology mentioned below.

(A) Sheaves and cosheaves: Sheaf theory helps in encoding how local data patches together globally. It is used in sensor networks, signal processing and integration of multi-model data. Further, cosheaf may be able to track how coverage of sensors overlaps and identifies blind spots, i.e., areas undetected in sensor overlap.

(B) Discrete Morse Theory: It is a combinatorial adaptation of Morse Theory. Let Y be a cw-complex and X be the set of all cells in Y. A function $\mu : X \to \mathbb{R}$ is said to be discrete Morse function if

(a) For any cell $\sigma \in X$, the number of cells $\tau \in X$ in the boundary of σ which satisfy $\mu(\sigma) \leq \mu(\tau)$ is at most one.

(b) For any $\sigma \in X$, the number of cells $\tau \in X$ containing σ in their boundaries, which satisfy $\mu(\sigma) \geq \mu(\tau)$ is at most one.

The Theory of discrete Morse function helps in simplifying simplicial complexes by collapsing cells, while preserving homotopy type and provides computational efficiency of persistent homology.

(C) Mapper Algorithms / Reeb graphs: It is a technique within TDA used to construct a simplified graphical representation of high dimensional data, revealing its underlying shape and patterns. This algorithm essentially approximates the Reeb graph of a data set, providing insights into its topological features such as loops, components and branches etc. It also offers a way to visualize and understand high dimensional data in a better interpretable graphical format. Note that Reeb graph represents how the level sets of a function evolve, merge, and split. It also allows multiscale analysis.

(D) Homotopy groups: They help in shaping classification of data up to homotopy, specially in robotic motion planning and in studying non-trivial loops in sensor networks.
(E) Cohomology and cup products: This tool helps not only in counting topological features and their nature but also captures interaction between different features.
(F) Spectral sequence: It relates homology at different scales or filtration and is used to study multi-parameter persistence (i.e., when more than one scale parameter is relevant, e.g., density and distance)
(G) Persistence combinatorial Laplacian: It is a new approach in data analysis that combines topological and geometric features. It extends persistence homology by using Laplacian operator (Δ or ∇^2 a second order differential operator calculating the divergence of the gradient of a function) to capture not only the topological features of a data set but also additional geometric and homotopic information that persistent homology misses. As seen in applications like protein engineering and image analysis, it allows far more powerful data analysis and machine learning tasks, such as classification and representation.
(H) Persistent Dirac operators: These are a class of mathematical tools used in TDA, particularly for representation and analysis of complex structures like molecular data. These are square roots of the persistent combinatorial Laplacian, which can be used to efficiently compute persistent Betti numbers, summarizing topological features at multiple scales. This allows to study the geometric structure of data also.

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CRYPTOSYSTEM IMPLEMENTING VERTEX MAGIC TOTAL LABELING

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Abstract. Cryptosystems are one of the most relevant and necessary aspects in today's world in order to facilitate the security and protection of information send through various communication channels. In this paper we introduce a new public key cryptosystem which uses vertex magic total labelling of graphs corresponding to the ciphertext for encryption and decryption procedures and uses RSA algorithm for the generation of public and private keys. We propose an algorithm for this and also illustrate the algorithm using an example.

1. Introduction

A graph G is an ordered pair (V, E) where V is the set of vertices and E is the set of edges. Graph labeling refers to assigning integers, real numbers or even objects to vertices, edges or both. Labeled graphs has many applications in coding theory[5], circuit design, communication networks[7], cryptography etc. In this paper, a graph labeling technique named vertex magic total labeling [4] is used for encryption.

The study of utilising mathematics to encrypt and decrypt data is known as cryptography. It is classified into two: symmetric-key cryptography and public key (asymmetric) cryptography. Public key cryptography is an asymmetric encryption method that encrypts data using a public key and decrypts it using a corresponding private key. The public key will be released into a public directory, where as the private key is kept secret. Hence only the person who knows the private key can decrypt the message. Using the

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Key words and phrases: Graph Labelling, Encryption, Decryption, Public Key, RSA

concept of public key cryptography Diffie and Hellman developed the algorithm known as RSA Algorithm[6], which is used in this paper to develop the public and private keys for our cryptosystem.

2. Preliminaries

Definition 2.1. A **graph labeling** [2] is an assignment of integers to the vertices or edges, or both, subject to certain conditions.

Definition 2.2. A **vertex magic total labeling** (VMTL) [3] of a graph $G = (V, E)$ is a bijection $f : V \cup E \to \{1, 2, 3......, |V| + |E|\}$ such that for every vertex v,

$$f(v) + \sum_{vw \epsilon E} f(vw) = k$$

where k is a constant.

Algorithm 2.3. Algorithm for Vertex Magic Total Labeling for K_n when n is odd [3]. For odd $n > 1$, we construct the labeling as follows. Consider an $n \times n$ matrix M. Let i, j be the indices for rows and columns respectively. The indices follow a zero based index. The process of filling the entries of the matrix is similar to constructing a magic square of order n. The only difference is that we stop filling the matrix once we have reached the number $n + \frac{n(n-1)}{2}$. The process is described below.

- **Step 1:** Let $x = 1$. Start populating the matrix from the position $j = \frac{n-1}{2}, i = \frac{n-1}{2}$
- **Step 2:** Fill the current (i, j) position with the value x, and increment x by 1.
- **Step 3:** The subsequent entries are filled by moving south-east by one position (i.e. $i = i + 1(mod n)$ and $j = j + 1(mod n)$) till a vacant entry is found. If the cell is already filled, then fill from the south-west direction (i.e. $i = i+1(mod n)$ and $j = j-1(mod n)$).The value of x has to be incremented after filling an entry. If x reaches $n + \frac{n(n-1)}{2}$ then proceed to step 4.
- **Step 4:** The remaining $n + \frac{n(n-1)}{2}$ entries are filled just by copying the non-diagonal entries by appealing to symmetry (i.e., just copying the numbers in the lower triangular matrix to the upper triangular matrix and vice-versa from filled cells to unfilled cells).

The vertex magic total labeling of K_7 is shown in the Figure 1.

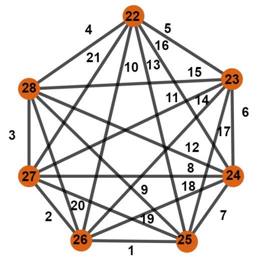

Figure 1. The VMTL of complete graph K_7

Remark 2.4. When n is even a similar algorithm is described by H.K. Krishnappa [3].

Definition 2.5. The **bifid cipher** is one of the classical cipher techniques that can be easily executed by hand. The bifid cipher uses square $n \times n$ grid and was invented by Felix Delastelle in 1901. In its simplest form it creates a grid and which maps the letters into numeric values[1]. All of the symbols needed to create the plaintext and ciphertext's alphabet can be found in this grid. In our cryptosystem we are going to use the 9×9 grid which is given in table 1.

	0	1	2	3	4	5	6	7	8
0	A	B	C	D	E	F	G	H	I
1	J	K	L	M	N	O	P	Q	R
2	S	T	U	V	W	X	Y	Z	0
3	1	2	3	4	5	6	7	8	9
4	a	b	c	d	e	f	g	h	i
5	j	k	l	m	n	o	p	q	r
6	s	t	u	v	w	x	y	z	.
7	,	:	;	?	!	"	'	-	*
8	(	)	[	]	{	}	$	=	%

Table 1. The 9×9 bifid cipher for the alphabet used

Remark 2.6. In order to get the numerical equivalent of a given text using bifid cipher, each symbol in the text is converted to a 2 digit number where the first digit represents the row and the second digit represents the column containing that symbol. For example the numerical equivalent of 'APPLE' using the bifid cipher in table 1 is 00 16 16 12 04.

3. The proposed VMTL cryptosystem

In this section we propose a new cryptosystem which uses the basics of the well known RSA cryptosystem [6] for key generation and an algorithm engaging VMTL for encryption and decryption processes.

Remark 3.1. In the following algorithm we use the term weight for the number assigned to edges and label for the number assigned to vertices.

Remark 3.2. Adjacency matrix in this section is defined as a matrix whose ij^{th}entry is the weight of the edge (v_i, v_j), when $i \neq j$ and is 0 when $i = j$.

3.1. **Key generation using RSA algorithm.**

(1) Select two prime numbers p and q (sufficiently large for practical purposes).
(2) Find $n = p \times q$.
(3) Calculate $\phi(n) = (p-1)(q-1)$.
(4) Choose an integer $1 < e < \phi(n)$, which is relatively prime to $\phi(n)$.
(5) Calculate d such that $de \equiv 1(mod\phi(n))$ (using Euclid's Division Algorithm).

(n, e) will be the public key and d will be the private key.

3.2. **Encryption Procedure.** Suppose we have to encrypt a message of length m.

(1) Add a unique letter say A at the beginning of the message (this letter will be fixed and predetermined for the cryptosystem).
(2) Find the numerical equivalent of the plaintext of length m using bifid cipher given in table 1. Let the numerical equivalent of the i^{th} character be denoted by p_i.
(3) Convert the message into a complete graph K_m with every character in the actual plaintext as a vertex labelled as $1, 2, 3, ..., m$. Add a vertex (labelled 0) representing the unique letter A and make it adjacent with the vertex representing the first character of the message. Let the resulting graph be G.
(4) A weight is assigned to each edge as the difference of the numerical equivalents of the end vertices of that edge. ($i.e.$, weight of $e_{ij} = p_i - p_j$, where $i > j$).
(5) Denote the adjacency matrix (as defined in remark 3.2) of this weighted graph as M_1'.

(6) Obtain the matrix M_1 by assigning 0, 1, 2, ..., m (*i.e.*, the order of occurrence in the plaintext of the letters represented by the vertices) as the diagonal entries of the matrix M_1'.
(7) Generate the vertex magic total labeling for K_{m+1} (*i.e.*, complete graph with letters in the plaintext and the unique letter A as vertices) using algorithm 2.3.
(8) Find its Hamiltonian cycle with minimum weight and denote its adjacency matrix as M_2.
(9) Multiply M_1 and M_2 to obtain the matrix C.
(10) Then obtain the matrix $M_2' = [M_{2_{ij}}']$ where $M_{2_{ij}}' \equiv (M_{2_{ij}})^e (mod\, n)$.
(11) As cipher text, send the matrix C as a single row (with space between each entry) and the matrix M_2'.

3.3. **Decryption Procedure.**

(1) Receiver calculates $M_2 = [M_{2_{ij}}]$ where $M_{2_{ij}} \equiv (M_{2_{ij}}')^d (mod\; n)$.
(2) Then M_1 is obtained as $M_1 = CM_2^{-1}$
(3) Construct the weighted graph G using the adjacency matrix M_1 and the unique letter.
(4) Message can be then decoded using table 1.

4. Illustration of the Cryptosystem

In this section we will illustrate the procedure described in the previous section by encrypting and decrypting the message ACTION.

Key generation:

(1) Select primes $p = 7$ and $q = 17$.
(2) $n = 7 \times 17 = 119$
(3) $\phi(n) = (7 - 1)(17 - 1) = 96$
(4) Choose an integer $e = 5$ which is relatively prime to $\phi(n) = 96$.
(5) Using Euclid's algorithm we get $d = 77$.
(6) (119, 5) can be released as the public key and 77 is kept as the private key.

Encryption:

(1) Here $m = 6$.
(2) We add the unique letter A at the beginning of the message and obtain the new message as AACTION.

(3) The numerical equivalent of the message AACTION is “00 00 02 21 08 15 14" which is obtained using the bifid cipher given in table 1. Then $p_0 = 00, p_1 = 00, p_2 = 02, \cdots$ and so on.

(4) Using steps 3 and 4 in encryption algorithm mentioned above, we obtain the weighted graph G as shown in figure 2.

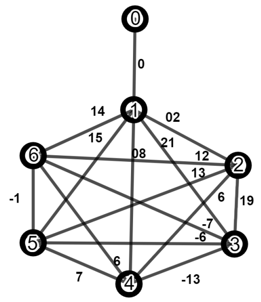

FIGURE 2. The weighted graph G mentioned in step 4 of the illustration

(5) Obtain the adjacency matrix of graph G in figure 2 as follows:

$$M_1' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 21 & 8 & 15 & 14 \\ 0 & 2 & 0 & 19 & 6 & 13 & 12 \\ 0 & 21 & 19 & 0 & -13 & -6 & -7 \\ 0 & 8 & 6 & -13 & 0 & 7 & 6 \\ 0 & 15 & 13 & -6 & 7 & 0 & -1 \\ 0 & 14 & 12 & -7 & 6 & -1 & 0 \end{bmatrix}$$

(6) Obtain M_1 by assigning the labels of the vertices (*i.e.*, 0, 1, 2, 3, 4, 5, 6) as diagonal entries of M_1'.

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 21 & 8 & 15 & 14 \\ 0 & 2 & 2 & 19 & 6 & 13 & 12 \\ 0 & 21 & 19 & 3 & -13 & -6 & -7 \\ 0 & 8 & 6 & -13 & 4 & 7 & 6 \\ 0 & 15 & 13 & -6 & 7 & 5 & -1 \\ 0 & 14 & 12 & -7 & 6 & -1 & 6 \end{bmatrix}$$

(7) Using vertex magic total labeling for K_7 (i.e., complete graph with the letters in the plaintext and the unique letter A as vertices) we

obtain the graph in figure 3.

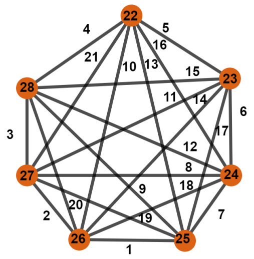

FIGURE 3. Graph K_7 obtained in step 7 after VMTL

(8) Obtain the adjacency matrix corresponding to the labelled graph K_7 as follows:

$$\begin{bmatrix} 22 & 5 & 16 & 13 & 10 & 21 & 4 \\ 5 & 23 & 6 & 17 & 14 & 11 & 15 \\ 16 & 6 & 24 & 7 & 18 & 8 & 12 \\ 13 & 17 & 7 & 25 & 1 & 19 & 9 \\ 10 & 14 & 18 & 1 & 26 & 2 & 20 \\ 21 & 11 & 8 & 19 & 2 & 27 & 3 \\ 4 & 15 & 12 & 9 & 20 & 3 & 28 \end{bmatrix}$$

(9) Figure 4 gives the Hamiltonian cycle in the graph in figure 3 with minimum weight.

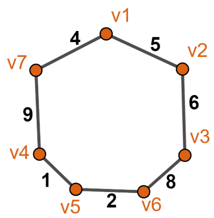

FIGURE 4. Minimum weight Hamiltonian cycle in K_7 with VMTL

(10) The adjacency matrix M_2 of the Hamiltonian cycle in figure 4 is given below.

$$M_2 = \begin{bmatrix} 0 & 5 & 0 & 0 & 0 & 0 & 4 \\ 5 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 8 & 0 & 2 & 0 & 0 \\ 4 & 0 & 0 & 9 & 0 & 0 & 0 \end{bmatrix}$$

(11) Compute the cipher text matrix C as:

$$C = M_1 M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 61 & 12 & 126 & 134 & 51 & 32 & 189 \\ 58 & 12 & 116 & 114 & 45 & 28 & 171 \\ 77 & 114 & 78 & -76 & -9 & 126 & 27 \\ 68 & 36 & 96 & 67 & -1 & 56 & -117 \\ 71 & 78 & 130 & -3 & 4 & 116 & -54 \\ 94 & 72 & 76 & 61 & -9 & 110 & -63 \end{bmatrix}$$

(12) Obtain M_2' where $M'_{2_{ij}} \equiv (M_{2_{ij}})^5 (mod\ 119)$

$$M_2' = \begin{bmatrix} 0 & 31 & 0 & 0 & 0 & 0 & 72 \\ 31 & 0 & 41 & 0 & 0 & 0 & 0 \\ 0 & 41 & 0 & 0 & 0 & 43 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 25 \\ 0 & 0 & 0 & 1 & 0 & 32 & 0 \\ 0 & 0 & 43 & 0 & 32 & 0 & 0 \\ 72 & 0 & 0 & 25 & 0 & 0 & 0 \end{bmatrix}$$

(13) Data to be send to the receiver is
0 0 0 0 0 0 0 61 12 126 134 51 32 189 58 12 116 114 45 28 171 77 114 78 -76 -9 126 27 68 36 96 67 -1 56 -117 71 78 130 -3 4 116 -54 94 72 76 61 -9 110 -63
together with the matrix M_2'.

Decryption:

(1) In the recipient side,we can calculate the matrix M_2 where $M_{2_{ij}} \equiv (M'_{2_{ij}})^{77} (mod\ 119)$.

$$M_2 = \begin{bmatrix} 0 & 5 & 0 & 0 & 0 & 0 & 4 \\ 5 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 8 & 0 & 2 & 0 & 0 \\ 4 & 0 & 0 & 9 & 0 & 0 & 0 \end{bmatrix}$$

(2) Then calculate M_1 as CM_2^{-1} :

$$M_1 = CM_2^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 21 & 8 & 15 & 14 \\ 0 & 2 & 2 & 19 & 6 & 13 & 12 \\ 0 & 21 & 19 & 3 & -13 & -6 & -7 \\ 0 & 8 & 6 & -13 & 4 & 7 & 6 \\ 0 & 15 & 13 & -6 & 7 & 5 & -1 \\ 0 & 14 & 12 & -7 & 6 & -1 & 6 \end{bmatrix}$$

From the adjacency matrix M_1 we can get the graph G given in figure 5.

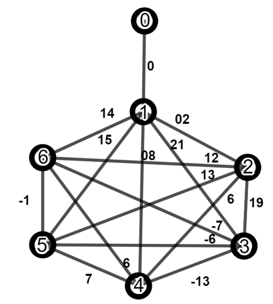

FIGURE 5. Graph G obtained from M_1

(3) We assume that the vertex labelled 0 is for A (which will be true always, as per our construction). Now we find the letters corresponding to the vertex i ($i = 1, 2, ..., 6$) using the relation "numerical equivalent of vertex $(i-1)$ + weight of edge joining vertices $(i-1)$ and (i) = numerical equivalent of vertex (i)" and using the table 1. Using this relation we get numerical equivalent of vertex 1 = 0 + 0 = 00, and from table 1 this represents letter A.
Numerical equivalent of vertex 2 = 2 + 0 = 02 which represents

letter C.
Numerical equivalent of vertex 3 = 19 + 2 = 21 which represents letter T.
Numerical equivalent of vertex 4 = -13 + 21 = 8 which represents letter I.
Numerical equivalent of vertex 5 = 7 + 8 = 15 which represents letter O.
Numerical equivalent of vertex 6 = -1 + 15 = 14 which represents letter N.
Thus we have decrypted the message as ACTION.

Remark 4.1. In this paper, we have used the Bifid cipher for numerical encoding to simplify the illustration. However, for practical applications, the Bifid cipher can be replaced with more modern and robust encoding schemes to create a stronger cryptosystem.

5. Complexity Analysis

In this section we present a complexity analysis of our algorithm using Big O notation, which is a method used to describe how efficient an algorithm is, particularly how its running time increases as the size of the input grows.

(1) In the proposed algorithm, first a vertex magic total labeling is to be given to a complete graph on n vertices. This VMTL has a complexity of $O(n^2)$ under optimized conditions.
(2) Two matrices operations are involved in the proposed algorithm of which adjacency matrix generation has a complexity of $O(n^2)$ and matrix multiplication of two matrices has standard time complexity of $O(n^3)$ using classical algorithm.
(3) In step 12, for an $m \times m$ matrix we are taking power of the entries and are reducing them modulo n, which has a complexity of $O(m^3 loge)$.
(4) In RSA key generation, if the prime numbers p and q are each of bit length n, then generating these large primes has a time complexity of $O(n^2)$, primarily due to the computational effort involved in identifying sufficiently large prime numbers. In contrast, Euclid's algorithm, has a time complexity of $O(n)$ which is relatively negligible compared to the other computational steps.

Concluding comments

In this paper, we propose a novel cryptosystem that employs vertex magic total labeling for both encryption and decryption processes. To enhance its practicality and security, the proposed cryptosystem can be further strengthened by integrating contemporary and more robust encoding techniques.

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CONVEX POLYGONS AND THE ISOPERIMETRIC PROBLEM IN SIMPLY CONNECTED SPACE FORMS M_κ^2

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Abstract. In this article, we prove that *there exists a unique perimeter minimizer among all piecewise smooth simple closed curves in M_κ^2 enclosing area* $A > 0$ ($A \leq 2\pi$ if $\kappa = 1$), *and it is a circle in M_κ^2 of radius* $AS_\kappa\left(\sqrt{A\,(4\pi - \kappa\, A)}/(2\pi)\right)$, where $AS_\kappa(t) := \begin{cases} t & \text{if } \kappa = 0, \\ \arcsin(t) & \text{if } \kappa = 1, \\ \sinh^{-1}(t) & \text{if } \kappa = -1. \end{cases}$

We also prove the isoperimetric inequality for M_κ^2. We give an elementary geometric proof which is uniform for all three simply connected space forms.

1. Introduction

Questions of the following type arise quite naturally. Why are small water droplets and bubbles that float in air approximately spherical? Why does a herd of reindeer form a circle if attacked by wolves? Of all geometric figures having certain property, which has greatest area or volume; and of all figures having certain property, which has least perimeter or surface area? These problems are capable of stimulating mathematical thought.

The isoperimetric problem on a surface is to enclose a given area with the shortest possible curve. The classical isoperimetric theorem asserts that in the Euclidean plane the unique solution is a circle. This property of the circle is most succinctly expressed in the form of an inequality called the isoperimetric inequality. The solution of isoperimetric problem for 'rectangles' was already known to Euclid. Little progress was made from Greek geometers until Swiss mathematicians Simon L'Huilier and Jacob Steiner

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of late eighteenth century. Using a symmetry argument Steiner has shown that the minimizer is a circle. However he did not prove the existence of a minimizer. By the use of 'approximating polygons', Edler filled this gap in 1882. However, these methods have long been forgotten and seem to have been rediscovered in [29]. Here, by analogous methods, we solve the isoperimetric problem on the simply connected surface M_κ^2 having constant sectional curvature κ ($\kappa = 0, \pm 1$), and prove that 'circle' is the unique solution to the isoperimetric problem. In this article, we give an elementary geometric proof which is uniform for all three simply connected space forms.

Before starting, a little more history is worth inserting. The history included here is taken mainly from the survey article of Osserman [37] which is about developments in the theory of isoperimetric inequalities. This survey recounts some of the most interesting of the many sharpened forms, various geometric versions, generalizations, and applications of this inequality. Also see the book by H. Hadwiger [28], Other general references given in [37] are Kazarinoff [30], Pόlya [[40], Chapter X], Porter [42], and the books of Blaschke listed in the bibliography. One aspect of the subject is given by Burago [15]. Others may be found in [38] and in the book of Santalò [47].

Most histories of the isoperimetric problem begin with its legendary origins in the "Problem of Queen Dido". Her problem (or at least one of them) was to enclose an optimal portion of land using a leather thong fashioned from oxhide. If Dido's was the true original isoperimetric problem, then what is needed is a solution not in the plane, but on a curved surface. (For more history of the classical case of curves in the plane see Mitrinović [36]). The consideration of the isoperimetric problem on curved surfaces goes quite a way back, at least to an 1842 paper of Steiner [50]. The fact that the smooth closed curve solving the isoperimetric problem on a surface must have constant geodesic curvature was mentioned in Steiner's paper [[50], p.150], and a proof was given in 1878 by Minding [35]. A detailed discussion is given in §18 of an extraordinary paper of Erhard Schmidt [49]. This paper provides an extended analysis of the isoperimetric problem on surfaces.

An interesting solution to the isoperimetric problem for curves on the sphere was given by F. Bernstein in 1905 [8]. A proof of the isoperimetric inequality for the hyperbolic plane was given in 1940 by Schmidt [[48], p.209].

Fiala [27] appears to have been the first to prove a general isoperimetric inequality for surfaces of variable Gauss curvature. See also Bol [13], Schmidt [[49], p.618], Aleksandrov [1], [[2], p.509] and Aleksandrov and Strel'cov [3, 4]. For the survey of the isoperimetric problem on general Riemannian manifolds refer to [[37], p.1211, §C].

The fact is, the isoperimetric inequality holds in the greatest generality imaginable, but one needs suitable definitions even to state it. The isoperimetric inequalities have proved useful in a number of problems in geometry, analysis, and physics.

We remark here that there are many other results of a similar nature, referred to as isoperimetric inequalities of mathematical physics, where extrema are sought for various quantities of physical significance such as the energy functional or the eigenvalues of a differential equation. They are shown to be extremal for a circular or spherical domain. Faber-Krahn Theorem [26, 32, 33] is an example of such results. Please see Rayleigh's fundamental treatise The theory of sound [[45], §210]. Extensive discussions of such problems can be found in the book of Pólya and Szegö [41] and the review article by Payne [39]. For some recent results of this type see [44, 31, 23, 24, 25]. For specific relations between the first non-zero eigenvalue of the Laplacian and geometric isoperimetric constants associated with compact Riemannian manifold, we refer to papers of Cheeger [22] and Yau [52]. (See also Buser [17, 18, 19, 20], Berger [5], Chavel [21] and Reilly [46]).

We now state the main results:

Theorem 1.1. *Fix $n \geq 3$ in $\mathbb{N}$ & $A \in \begin{cases} (0,\infty) & \text{if } \kappa = 0, \\ (0, 2\pi) & \text{if } \kappa = 1, \\ (0, (n-2)\,\pi) & \text{if } \kappa = -1. \end{cases}$ Among all polygons with n sides in M_κ^2 having area A, perimeter minimizer is the regular n-gon.*

$$\text{Let } AS_\kappa(t) := \begin{cases} t & \text{if } \kappa = 0, \\ \arcsin(t) & \text{if } \kappa = 1, \\ \sinh^{-1}(t) & \text{if } \kappa = -1. \end{cases}$$

Theorem 1.2. *Fix $A > 0$ ($A \leq 2\pi$ if $\kappa = 1$). There exists a unique perimeter minimizer among all piecewise smooth simple closed curves in M_κ^2 enclosing area A, and it is a circle in M_κ^2 of radius $AS_\kappa\left(\sqrt{A\,(4\pi - \kappa\,A)}/(2\pi)\right)$.*

Corollary 1.3. *Fix $A > 0$ ($A \leq 2\pi$ if $\kappa = 1$). There exists a unique perimeter minimizer among all piecewise smooth simple closed curves in M_κ^2 having m components each enclosing area $A_i > 0$ such that $A = \sum_{i=1}^m A_i$, and it is a circle in M_κ^2 of radius $AS_\kappa\left(\sqrt{A\,(4\pi - \kappa\,A)}/(2\pi)\right)$.*

Theorem 1.4 (The Isoperimetric Inequality for M_κ^2)**.** *For any piecewise smooth simple closed curve $\mathcal{C}$ in M_κ^2 with arc-length ℓ and enclosing area $A > 0$ ($A \leq 2\pi$ if $\kappa = 1$) we have $\ell^2 \geq 4\pi A - \kappa\,A^2$ and equality holds if and only if $\mathcal{C}$ is a circle in M_κ^2 of radius $AS_\kappa\left(\sqrt{A\,(4\pi - \kappa\,A)}/(2\pi)\right)$.*

In section 2, we introduce the model spaces M_κ^2 (as Riemannian manifolds) and discuss isometries of M_κ^2. In section 3, we state few results on triangles and polygons in M_κ^2 and we have given proofs mostly when the results are not available in books. Regular polygons in M_κ^2 are studied in section 4. Section 5 contains the proof of Theorem 1.1. In section 6, proofs of Theorem 1.2, Corollary 1.3, Theorem 1.4 are given. Section 7 is an appendix to this article.

2. ISOMETRIES OF M_κ^2

A space form is a complete Riemannian manifold with constant sectional curvature κ. Complete, simply connected Riemannian manifolds of dimension d, with constant sectional curvature κ are denoted by M_κ^d.

Let $< \, , \, >_0$ denote the standard inner product of the Euclidean space $\mathbb{E}^d$ ($d \in \mathbb{N}$). The Euclidean space $(\mathbb{E}^2, < \, , \, >_0)$ and $S^2 = \{x \in \mathbb{E}^3 \mid < x, x >_0 = 1\}$, the unit sphere in $\mathbb{E}^3$ with induced Riemannian metric from $\mathbb{E}^3$ are the model spaces for M_0^2 and M_1^2 respectively. The hyperboloid of one sheet $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|x_1^2 + x_2^2 - x_3^2 = -1 \;\&\; x_3 > 0\}$ with the Riemannian metric induced from the bilinear form $\langle x, y\rangle_{-1} := x_1\,y_1 + x_2\,y_2 - x_3\,y_3$ where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ is the model space for M_{-1}^2. The *inner metric* d_κ of M_κ^2 is given by the formula

$$d_\kappa(x,y) = \begin{cases} \sqrt{\langle x-y, x-y\rangle_0} & \text{if } \kappa = 0 \\ AC_\kappa\,(\kappa\, < x, y >_\kappa) & \text{if } \kappa \neq 0 \end{cases} \quad \forall\, x, y \in M_\kappa^2,$$

where

$$AC_\kappa(t) := \begin{cases} t & \text{if } \kappa = 0, \\ \arccos(t) & \text{if } \kappa = 1, \\ \operatorname{arccosh}(t) & \text{if } \kappa = -1. \end{cases}$$

Please note that if (M,g) is a Riemannian manifold with sectional curvature κ. Then for any constant $c>0$, the sectional curvature of (M,cg) equals $\frac{\kappa}{c}$. This is the reason why we can rescale the metric and consider only the cases $\kappa=-1,0,1$.

For $p\in M_\kappa^2$ and $r>0$ ($r<\pi$ if $\kappa=1$), $B_\kappa(p,r):=\{x\in M_\kappa^2 \mid d_\kappa(p,x)<r\}$ denotes the open ball in M_κ^2 with center p and radius r. Its boundary is $\mathcal{C}_\kappa(p,r):=\{x\in M_\kappa^2 \mid d_\kappa(p,x)=r\}$.

If we take

$$p_0=\begin{cases}(0,0) & \text{if } \kappa=0\\ (0,0,1) & \text{if } \kappa\neq 0\end{cases} \tag{1}$$

then $\mathcal{C}_\kappa(p_0,r)$ is nothing but a Euclidean circle in the plane

$$\{(x_1,x_2,|\kappa|\,C_\kappa(r)) \mid x_1,x_2\in\mathbb{R}\}\subseteq\mathbb{R}^3$$

with center $C_\kappa(r)\,p_0$ and radius $S_\kappa(r)$, where

$$C_\kappa(t)=\begin{cases}t & \text{if } \kappa=0\\ \cos t & \text{if } \kappa=1\\ \cosh t & \text{if } \kappa=-1\end{cases} \quad\text{and}\quad S_\kappa(t)=\begin{cases}t & \text{if } \kappa=0\\ \sin t & \text{if } \kappa=1\\ \sinh t & \text{if } \kappa=-1.\end{cases}$$

We say that $\mathcal{C}_\kappa(p,r)$ is a *circle* in M_κ^2 of radius r. The area of the ball $B_\kappa(p,r)$ is $4\,\pi\,S_\kappa^2\left(\frac{r}{2}\right)$. The perimeter of the ball $B_\kappa(p,r)$ is $2\,\pi\,S_\kappa\,(r)$. Please note here that the model space $\mathbb{E}^2$ is identified with the x_1x_2-plane in $\mathbb{R}^3$.

Let $\tilde{H}_0$ denote a line in $\mathbb{E}^2$. Let $\tilde{H}_\kappa$, $\kappa\neq 0$, denote a 2-dimensional vector subspace of $\mathbb{R}^3$. Let n_κ be a unit vector normal to $\tilde{H}_\kappa$ at any point of $\tilde{H}_\kappa$. Let $H_\kappa:=\tilde{H}_\kappa\cap M_\kappa^2$. We call H_κ a *line* in M_κ^2. Then $M_\kappa^2\setminus H_\kappa$ has two connected components. We call these components having H_κ as common boundary as *open half-spaces* in M_κ^2. When $\kappa=1$ they are the open hemispheres in S^2.

Definition 2.1. The *reflection* r_{H_κ} through a *line* H_κ in M_κ^2 is defined as

$$r_{H_\kappa}(x)=x-2<x,n_\kappa>_\kappa\,n_\kappa.$$

Definition 2.2. Let (M,g) be a Riemannian manifold. A diffeomorphism $\varphi:M\to M$ is called an *isometry* of (M,g) if the differential $d\varphi$ preserves

Riemannian metric, i.e., for all $x \in M$ and for all pairs $u, v \in T_xM$ we have

$$g_x(u,v) = g_{\varphi(x)}(d\varphi|_x(u), d\varphi|_x(v)).$$

Remark 2.3. Any isometry φ of (M, g) satisfies $d(\varphi(x), \varphi(y)) = d(x, y)\ \forall\, x, y \in M$, where d is the inner metric of (M, g).

Proposition 2.4. *Given any positive integer k and two sets of k points $\{A_1, \dots, A_k\}$ and $\{B_1, \dots, B_k\}$ in M^2_κ such that $d_\kappa(A_i, A_j) = d_\kappa(B_i, B_j)$ $\forall\, i, j \in \{1, \dots, k\}$ there exists an isometry of M^2_κ mapping A_i to B_i $\forall\, i \in \{1, \dots, k\}$. Moreover, one can obtain such an isometry by composing k or fewer reflections through lines.*

(cf. Proposition 2.17 of [14])

Proposition 2.5. *Let ϕ be an isometry of M^2_κ.*

(1) *If ϕ is not the identity, then the set of points which it fixes is contained in a line.*
(2) *If ϕ acts as the identity on some line H_κ, then ϕ is either the identity or the reflection r_{H_κ} through the line H_κ.*
(3) *ϕ can be written as the composition of three or fewer reflections through lines.*

(cf. Proposition 2.18 of [14])

We now describe the Isometry group of the model spaces M^2_κ, denoted as $\mathrm{Iso}(M^2_\kappa)$. Let $O(d)$, $d \in \mathbb{N}$, denote the group of orthogonal real matrices, i.e., those real $d \times d$ matrices A which satisfy ${}^tA\,A = \mathrm{Id}$, where ${}^tA$ is the transpose of A and Id is the identity matrix. Consider the group $GL(d+1, \mathbb{R})$ (thought of as matrices) with the usual linear action on $\mathbb{R}^{d+1}$. Let $O(d,1)$ denote the subgroup of $GL(d+1,\mathbb{R})$ consisting of those matrices which leave invariant the bilinear form $< \cdot, \cdot >_{-1}$. A simple calculation shows that $O(d,1)$ consists of those $(d+1) \times (d+1)$ matrices A such that ${}^tA\,J\,A = J$, where J is the diagonal matrix with entries $(1, 1, \dots, 1, -1)$ in the diagonal. Let $O(d,1)_+ \subseteq O(d,1)$ be the subgroup consisting of those matrices in $O(d,1)$ whose bottom right hand entry is positive.

Proposition 2.6. (i) *Iso(M^2_0)$\cong \mathbb{R}^2 \rtimes O(2)$, the semi direct product of additive group of $\mathbb{R}^2$ with the orthogonal group $O(2)$.*
(ii) *Iso(M^2_1) $\cong O(3)$.*
(iii) *Iso(M^2_{-1}) $\cong O(2,1)_+$.*

(cf. Theorem 2.24 of [14])

3. GEODESIC SEGMENTS, TRIANGLES AND POLYGONS IN M_κ^2

Definition 3.1. Connected subsets of line H_κ in M_κ^2 are called *geodesic segments* of M_κ^2.

Consider $x, y \in M_\kappa^2$ such that $x \neq y$ ($x \neq \pm y$ when $\kappa = 1$). Put $v := y - \kappa < y, x >_\kappa x + (|\kappa| - 1)x$. Then, $v \in T_x M_\kappa^2$. We denote

$$\left\{ C_\kappa(t)^{|\kappa|}\, x + S_\kappa(t)\, \frac{v}{\sqrt{< v, v >_\kappa}} \;\middle|\; 0 \leq t \leq d_\kappa(x, y) \right\} \text{ by } [x, y].$$

Then $[x, y]$ is a geodesic segment in M_κ^2 joining x and y.

For $p \in M_\kappa^2$ and unit vector $v \in T_p M_\kappa^2 \setminus \{\underline{0}\}$, let $\gamma_{p,v}$ denote the geodesic with the initial conditions $\gamma_{p,v}(0) = p$ and $\gamma'_{p,v}(0) = v$. Then

$$\gamma_{p,v}(t) = C_\kappa(t)^{|\kappa|}\, p + S_\kappa(t)\, \frac{v}{\sqrt{< v, v >_\kappa}} \quad (t \in \mathbb{R}).$$

Every geodesic $\gamma_{p,v}$ with given $p \in M_\kappa^2$ and $v \in T_p M_\kappa^2$ is a unit speed parametrisation of a unique line H_κ such that $p \in H_\kappa$, $v \in \tilde{H}_\kappa$. A *polygon* $\wp$ in M_κ^2 is a closed region whose boundary $\partial\wp$ is a simple closed curve (i.e., it is homeomorphic to S^1) consisting of geodesic segments. A point p of $\partial\wp$ is called a *vertex* of $\wp$ if $\partial\wp$ intersected with some disc with center p consists of two radial geodesic segments which are not extensions of each other. The geodesic segments constituting $\partial\wp$ are called *sides* of $\wp$. For a vertex p of a polygon $\wp$, let γ_{p,v_1} and γ_{p,v_2} denote the sides of $\wp$ having common vertex p. A positive orientation of $\partial\wp$ with respect to the bounded domain $\overset{\circ}{\wp}$ is to traverse $\partial\wp$ in a direction such that $\overset{\circ}{\wp}$ lies to the left of $\partial\wp$ while moving along it. Let q_1, q_2 be the end points of $\gamma_{p,v_1}, \gamma_{p,v_2}$ respectively. If the successive points q_1, p, q_2 represent the positive orientation of the boundary then, the *angle* of polygon $\wp$ at vertex p is defined as

$$\measuredangle \text{ at } p := \begin{cases} \measuredangle\{v_1, v_2\} & \text{if } \det(v_1, v_2, n(p)) < 0, \\ 2\pi - \measuredangle\{v_1, v_2\} & \text{if } \det(v_1, v_2, n(p)) > 0. \end{cases}$$

Here, $n(p)$ is the unit normal to M_κ^2 at $p := (p_1, p_2, p_3) \in M_\kappa^2 \subset \mathbb{R}^3$. That is,

$$n(p) = \begin{cases} e_3 & \text{if } \kappa = 0, \\ p & \text{if } \kappa = 1, \\ (p_1, p_2, -p_3) & \text{if } \kappa = -1. \end{cases}$$

A polygon $\wp$ is said to be *convex* if for any $x, y \in \wp$ (with $y \neq -x$ if $\kappa = 1$), the geodesic segment $[x, y]$ is contained in $\wp$. A polygon $\wp$ is said to be *locally convex* if for any $x \in \wp$, $B_\kappa(x, r) \cap \wp$ is convex $\forall\ r > 0$. Note that a connected locally convex polygon is convex and vice versa. A polygon in M_1^2 is called *proper polygon* if it contains no pair of antipodal points. A polygon (proper polygon if $\kappa = 1$) of n sides is called an *n-gon* in M_κ^2. Note that for any n-gon, $n \geq 3$ always holds. For $\kappa \neq 1$, any 3-gon is always convex. A convex 3-gon in M_κ^2 is called a *triangle* in M_κ^2. A triangle in M_κ^2 having vertices x, y, $z \in M_\kappa^2$ is denoted by $[x, y, z]$.

Law of Cosine for triangles in M_κ^2:

$\underline{\kappa = 0}$ $\quad c^2 = a^2 + b^2 - 2\,a\,b\,\cos\gamma$,

$\underline{\kappa \neq 0}$ $\quad C_\kappa(c) = C_\kappa(a)\ C_\kappa(b) + \kappa\, S_\kappa(a)\ S_\kappa(b)\ \cos\gamma$,

where a, b, c are the sides of the triangle and γ is the angle opposite to side c.

In particular, fixing a, b and κ, one sees that c is a strictly increasing function of $\gamma \in [0, \pi]$. The triangle inequality for a triangle in M_κ^2 follows from the Law of Cosine. Strict triangle inequality holds for triangles in M_κ^2.

Law of Sine for triangles in M_κ^2: $\quad \dfrac{S_\kappa(a)}{\sin\alpha} = \dfrac{S_\kappa(b)}{\sin\beta} = \dfrac{S_\kappa(c)}{\sin\gamma}$,
where a, b, c are the sides of the triangle and α, β, γ are the angles opposite to sides a, b, c respectively.

Theorem 3.2. *The area of a triangle T in M_κ^2 ($\kappa \neq 0$) with angles α, β, γ is equal to $\kappa\ (\alpha + \beta + \gamma - \pi)$.*

Proof. By Gauss-Bonnet Formula, $(\alpha + \beta + \gamma - \pi)$ is nothing but $\int_T \kappa\, dV$, where dV is the area element of M_κ^2. Therefore, for M_κ^2 ($\kappa \neq 0$), area of triangle T is equal to $\kappa\ (\alpha + \beta + \gamma - \pi)$. ☐

Remark 3.3.
(1) For $\kappa = 1$, Theorem 3.2 is known as *Girard's Theorem.*
(2) The area of a triangle in $\mathbb{E}^2$ can not be determined only from its three angles.
(3) The area of the disk $B := B_{-1}\left(p, 2\,\sinh^{-1}\left(\frac{1}{2}\right)\right)$, $p \in M_{-1}^2$, is π which is greater than area of any triangle in M_{-1}^2. Hence there is no triangle in M_{-1}^2 which can inscribe the disk B. **Triangles in $\boldsymbol{M_{-1}^2}$ are thin!**

Theorem 3.4. *The area A of a triangle in M_κ^2 with sides a, b, c is given by the equation*

$$T_{|\kappa|}(A/4) = \sqrt{T_\kappa(s/2)\, T_\kappa\left[(s-a)/2\right]\, T_\kappa\left[(s-b)/2\right]\, T_\kappa\left[(s-c)/2\right]} \qquad (2)$$

where $s := (a+b+c)/2$ *and* $T_\kappa(t) := \begin{cases} t & \text{if } \kappa = 0, \\ \tan t & \text{if } \kappa = 1, \\ \tanh t & \text{if } \kappa = -1. \end{cases}$

Proof. $\underline{\kappa = 0}$**:** Let γ be the angle included between the sides a and b. From the Law of Cosine we have

$$\cos\gamma = \frac{a^2+b^2-c^2}{2\,a\,b}.$$

Hence, $\sin\gamma = \sqrt{1-\cos^2\gamma} = \dfrac{2}{a\,b}\sqrt{s\,(s-a)\,(s-b)\,(s-c)}$ (and the Law of Sine follows immediately). Therefore,

$$A = \frac{1}{2}\,a\,b\,\sin\gamma = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}.$$

$\underline{\kappa \neq 0}$**:** In what follows the equations (A-1), (A-2), (A-3), ... and (B-1), (B-2), (B-3), ... refer to equations from Appendix A and Appendix B respectively which appear at the end of the article. By Theorem 3.2,

$$\begin{aligned}
\tan\left(\frac{A}{4}\right) &= \tan\left(\frac{\kappa\,(\alpha+\beta+\gamma-\pi)}{4}\right) = \frac{\sin\left(\frac{\kappa\,(\alpha+\beta+\gamma-\pi)}{4}\right)}{\cos\left(\frac{\kappa\,(\alpha+\beta+\gamma-\pi)}{4}\right)} \\
&= \kappa\,\frac{\sin\left(\frac{\alpha+\beta+\gamma-\pi}{4}\right)}{\cos\left(\frac{\alpha+\beta+\gamma-\pi}{4}\right)} \text{ [by (A-1)]} \\
&= \kappa\,\frac{\sin\left(\frac{\alpha+\beta}{2}\right) - \sin\left(\frac{\pi-\gamma}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\pi-\gamma}{2}\right)} \text{ [by (A-14) and (A-15)]} \\
&= \kappa\,\frac{\sin\left(\frac{\alpha+\beta}{2}\right) - \cos\frac{\gamma}{2}}{\cos\left(\frac{\alpha+\beta}{2}\right) + \sin\frac{\gamma}{2}} \text{ [by (A-3) and (A-7)]} \\
&= \kappa\,\frac{\left[\frac{C_\kappa\left(\frac{a-b}{2}\right)}{C_\kappa\left(\frac{c}{2}\right)} - 1\right]\cos\frac{\gamma}{2}}{\left[\frac{C_\kappa\left(\frac{a+b}{2}\right)}{C_\kappa\left(\frac{c}{2}\right)} + 1\right]\sin\frac{\gamma}{2}} \qquad \text{[by (B-4) and (B-6)].}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tan\left(\frac{A}{4}\right) &= \kappa\,\frac{C_\kappa\left(\frac{a-b}{2}\right)-C_\kappa\left(\frac{c}{2}\right)}{C_\kappa\left(\frac{a+b}{2}\right)+C_\kappa\left(\frac{c}{2}\right)}\cdot\frac{\cos\frac{\gamma}{2}}{\sin\frac{\gamma}{2}}=\frac{S_\kappa\left(\frac{s-a}{2}\right)S_\kappa\left(\frac{s-b}{2}\right)}{C_\kappa\left(\frac{s}{2}\right)C_\kappa\left(\frac{s-c}{2}\right)}\cdot\frac{\cos\frac{\gamma}{2}}{\sin\frac{\gamma}{2}}\\
&\quad\text{[by (A-15), (A-16) and (A-1)]}\\
&= \frac{S_\kappa\left(\frac{s-a}{2}\right)S_\kappa\left(\frac{s-b}{2}\right)}{C_\kappa\left(\frac{s}{2}\right)C_\kappa\left(\frac{s-c}{2}\right)}\sqrt{\frac{S_\kappa(s)\,S_\kappa(s-c)}{S_\kappa(s-a)\,S_\kappa(s-b)}}\ \text{[by (B-1), (B-2)]}\\
&= \frac{S_\kappa\left(\frac{s-a}{2}\right)S_\kappa\left(\frac{s-b}{2}\right)}{C_\kappa\left(\frac{s}{2}\right)C_\kappa\left(\frac{s-c}{2}\right)}\sqrt{\frac{S_\kappa\left(\frac{s}{2}\right)C_\kappa\left(\frac{s}{2}\right)S_\kappa\left(\frac{s-c}{2}\right)C_\kappa\left(\frac{s-c}{2}\right)}{S_\kappa\left(\frac{s-a}{2}\right)C_\kappa\left(\frac{s-a}{2}\right)S_\kappa\left(\frac{s-b}{2}\right)C_\kappa\left(\frac{s-b}{2}\right)}}\\
&\quad\text{[by (A-4)]}\\
&= \sqrt{T_\kappa\left(\frac{s}{2}\right)T_\kappa\left(\frac{s-a}{2}\right)T_\kappa\left(\frac{s-b}{2}\right)T_\kappa\left(\frac{s-c}{2}\right)}.
\end{aligned}$$

□

Remark 3.5. Equation (2) is known as *Heron's formula* and *L'Huilier's formula* for $\kappa = 0$ and $\kappa = 1$ respectively.

Proposition 3.6. *Given two sides a, b and the included angle γ of a triangle in M^2_κ, its area A is given by the formula*

$$CT_{|\kappa|}(A/2) = \frac{CT_\kappa(a/2)\;CT_\kappa(b/2)\;(\sin^2\gamma)^{1-|\kappa|}+\kappa\,\cos\gamma}{\sin\gamma},$$

where $CT_\kappa(t) := \begin{cases} t & \text{if } \kappa = 0,\\ \cot t & \text{if } \kappa = 1,\\ \coth t & \text{if } \kappa = -1.\end{cases}$

Proof. $\underline{\kappa = 0}$**:** $\quad A = \dfrac{1}{2}\,a\,b\,\sin\gamma.$
$\underline{\kappa \neq 0}$**:** Let α, β be the other two angles of the triangle opposite to sides a, b respectively. By Theorem 3.2,

$$\begin{aligned}
\sin\frac{A}{2} &= \sin\left(\frac{\kappa\,(\alpha+\beta+\gamma-\pi)}{2}\right) = \kappa\,\sin\left(\frac{\alpha+\beta+\gamma-\pi}{2}\right)\quad\text{[by (A-1)]}\\
&= -\kappa\,\cos\left(\frac{\alpha+\beta+\gamma}{2}\right)\quad\text{[by (A-3)]}\\
&= -\kappa\left[\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)-\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\right]\quad\text{[by (A-6)]}.
\end{aligned}$$

Thus, using (B-4) and (B-6), we get

$$\sin\frac{A}{2} = -\kappa\,\frac{C_\kappa\left(\dfrac{a+b}{2}\right)-C_\kappa\left(\dfrac{a-b}{2}\right)}{C_\kappa\left(\dfrac{c}{2}\right)}\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)$$

Now, using (A-4) and (A-16) we infer that,

$$\sin\left(\frac{A}{2}\right)=\frac{S_\kappa\left(\dfrac{a}{2}\right)S_\kappa\left(\dfrac{b}{2}\right)\sin\gamma}{C_\kappa\left(\dfrac{c}{2}\right)}. \tag{3}$$

$$\begin{aligned}
\cos\left(\frac{A}{2}\right) &= \cos\left(\frac{\kappa\,(\alpha+\beta+\gamma-\pi)}{2}\right)=\cos\left(\frac{\alpha+\beta+\gamma-\pi}{2}\right) \quad \text{[by (A-1)]}\\
&= \sin\left(\frac{\alpha+\beta+\gamma}{2}\right) \quad \text{[by (A-7)]}\\
&= \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) \quad \text{[by (A-2)]}\\
&= \frac{C_\kappa\left(\dfrac{a-b}{2}\right)}{C_\kappa\left(\dfrac{c}{2}\right)}\cos^2\left(\frac{\gamma}{2}\right)+\frac{C_\kappa\left(\dfrac{a+b}{2}\right)}{C_\kappa\left(\dfrac{c}{2}\right)}\sin^2\left(\frac{\gamma}{2}\right)\\
&\qquad\qquad \text{[by (B-4) and (B-6)]}\\
&= \frac{\cos^2\left(\frac{\gamma}{2}\right)\left[C_\kappa\left(\frac{a}{2}\right)C_\kappa\left(\frac{b}{2}\right)+\kappa\,S_\kappa\left(\frac{a}{2}\right)S_\kappa\left(\frac{b}{2}\right)\right]}{C_\kappa\left(\frac{c}{2}\right)} \quad \text{[by (A-7)]}\\
&\quad+\frac{\sin^2\left(\frac{\gamma}{2}\right)\left[C_\kappa\left(\frac{a}{2}\right)C_\kappa\left(\frac{b}{2}\right)-\kappa\,S_\kappa\left(\frac{a}{2}\right)S_\kappa\left(\frac{b}{2}\right)\right]}{C_\kappa\left(\frac{c}{2}\right)} \quad \text{[by (A-6)]}\\
&= \frac{C_\kappa\left(\frac{a}{2}\right)C_\kappa\left(\frac{b}{2}\right)+\kappa\,S_\kappa\left(\frac{a}{2}\right)S_\kappa\left(\frac{b}{2}\right)\left[\cos^2\left(\frac{\gamma}{2}\right)-\sin^2\left(\frac{\gamma}{2}\right)\right]}{C_\kappa\left(\frac{c}{2}\right)}.
\end{aligned}$$

Hence by (A-8),

$$\cos\frac{A}{2}=\frac{C_\kappa\left(\frac{a}{2}\right)C_\kappa\left(\frac{b}{2}\right)+\kappa\,S_\kappa\left(\frac{a}{2}\right)S_\kappa\left(\frac{b}{2}\right)\cos\gamma}{C_\kappa\left(\frac{c}{2}\right)}. \tag{4}$$

From (3) and (4) we get, $\cot\left(\dfrac{A}{2}\right) = \dfrac{CT_\kappa\left(\dfrac{a}{2}\right)\, CT_\kappa\left(\dfrac{b}{2}\right) + \kappa\, \cos\gamma}{\sin\gamma}$. □

Definition 3.7. Let $T := [P, Q, R]$, $T' := [P', Q', R']$ be triangles in M_κ^2. We say that the triangle T is *congruent* to T' if there exists an isometry f of M_κ^2 such that $f(P) = P'$, $f(Q) = Q'$ and $f(R) = R'$.

Proposition 3.8. *Let T, T' be triangles in M_κ^2. Let a, b, c (resp. a', b', c') be the sides of T (resp. T'). Let α, β, γ be angles of T opposite to sides a, b, c respectively. Let α', β', γ' be angles of T' opposite to sides a', b', c' respectively. Then, the following are equivalent:*

(i) *T is congruent to T'.*
(ii) $a = a'$, $b = b'$, $c = c'$.
(iii) $\alpha = \alpha'$, $b = b'$, $c = c'$.
(iv) $a = a'$, $\beta = \beta'$, $c = c'$.

Each of the above imply

(v) $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$.

For $\kappa \neq 0$, all the five statements above are equivalent.

Proof. See Appendix C for a proof of this Proposition. □

Proposition 3.9. *Among all triangles in M_κ^2 whose two sides are of length a, b ($a + b < \pi$ if $\kappa = 1$), area maximizer is the triangle whose vertices lie on a circle having the midpoint of the 'remaining side' as its center.*

Proof. Let $\mathcal{U}$ denote the family of all triangles in M_κ^2 whose two sides are of length a, b ($a + b < \pi$ if $\kappa = 1$). Let $r_0 := a + b$.
Existence Upto congruence all triangles in $\mathcal{U}$ lie inside $B_\kappa(p, r_0)$ where $p \in M_\kappa^2$. Since $\overline{B_\kappa(p, r_0)}$ is compact in (M_κ^2, d_κ) there exists an 'area maximizer' T_0 in $\mathcal{U}$.
Let $\gamma := \gamma(T)$ be the angle of a triangle T in $\mathcal{U}$ included between the sides of length a, b. Put $A_\gamma = area(T)$. By Proposition 3.6, when $\kappa = 0$

$$A_\gamma = \frac{1}{2}\, a\, b\, \sin\gamma \leq \frac{1}{2}\, a\, b\, \sin\frac{\pi}{2} = A_{\frac{\pi}{2}}.$$

So, $area(T)$ is maximum when $\gamma = \dfrac{\pi}{2}$.
Consider $\kappa \neq 0$. By Proposition 3.6,

$$\cot(A_\gamma/2) = \frac{CT_\kappa(a/2)\; CT_\kappa(b/2) + \kappa\, \cos\gamma}{\sin\gamma}. \tag{5}$$

Consider the unit circle S^1 in $\mathbb{E}^2$ with center at $(0,0) =: O$. Let $Q = Q(\gamma)$ be a point in $\mathbb{E}^2$ such that $\| Q - O \|_{\mathbb{E}^2} = CT_\kappa(a/2)\, CT_\kappa(b/2)$. The information on a, b implies that $CT_\kappa(a/2)\, CT_\kappa(b/2) > 1$, and hence Q lies 'outside' S^1. Extend the line segment $[Q, O]$ and intersect S^1 at R. Let $P = P(\gamma)$ be the point on S^1 such that $\measuredangle POR = (1-\kappa)\,\dfrac{\pi}{2} + \kappa\,\gamma$. Let N be the orthogonal projection of P on the line joining Q & R. Then,

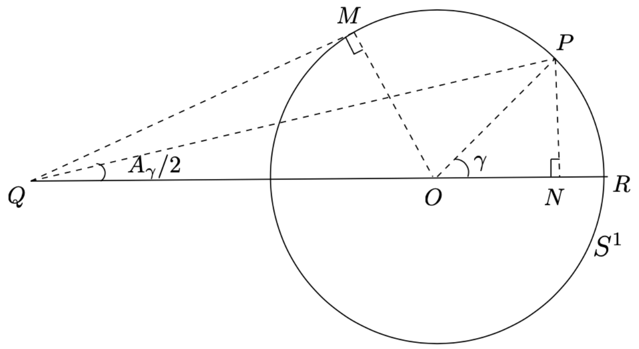

Fig. 1

$$\| Q - N \|_{\mathbb{E}^2} = \begin{cases} \| Q - O \|_{\mathbb{E}^2} + \kappa \; \| N - O \|_{\mathbb{E}^2} & \text{if } \gamma \in (0, \frac{\pi}{2}], \\ \| Q - O \|_{\mathbb{E}^2} - \kappa \; \| N - O \|_{\mathbb{E}^2} & \text{if } \gamma \in [\frac{\pi}{2}, \pi). \end{cases}$$

Therefore,

$$\begin{aligned} \| Q - N \|_{\mathbb{E}^2} &= CT_\kappa(a/2)\, CT_\kappa(b/2) + \cos\Big((1-\kappa)\,\frac{\pi}{2} + \kappa\,\gamma\Big) \\ &= CT_\kappa(a/2)\, CT_\kappa(b/2) + \kappa\, \cos\gamma. \end{aligned}$$

By (5),

$$\measuredangle PQR = A/2\,. \tag{6}$$

Let $M = M(\gamma)$ be the point on S^1 such that line QM is tangent to S^1 at M and M lies on the same side of line QR as P. Then $\measuredangle PQR$ is maximum when $P = M$. Hence, it follows by (6) that $P(\gamma_0) = M(\gamma_0)$ where $\gamma_0 := \gamma(T_0)$. Recall here that T_0 is an area maximizer in $\mathcal{U}$. Thus, $\measuredangle POR$ being an external angle of a triangle in $\mathbb{E}^2$,

$$(1-\kappa)\frac{\pi}{2} + \kappa\gamma_0 = \measuredangle P(\gamma_0)OR = \measuredangle M(\gamma_0)OR = \frac{\pi}{2} + \measuredangle M(\gamma_0)QO = \frac{\pi}{2} + \frac{A_0}{2} \tag{7}$$

where $A_0 := area(T_0)$.

Let α_0, β_0 be the angles of T_0 other than γ_0. As $A_0 = \kappa\,(\alpha_0 + \beta_0 + \gamma_0 - \pi)$, (7) implies that $\gamma_0 = \alpha_0 + \beta_0$. Let A, B, C be the vertices of T_0 having angles α_0, β_0, γ_0 respectively. As $\gamma_0 > \alpha_0$, there is a unique point D on the side $[A, B]$ of T_0 such that $\measuredangle ACD = \alpha_0$. Then $\measuredangle BCD = \beta_0$. Thus the triangles $[A, D, C]$ & $[B, D, C]$ are isosceles triangles. Hence the geodesic segments $[A, D]$, $[D, C]$, $[D, B]$ are all of same length. Thus the vertices of T_0 lie on a circle whose center is the midpoint of side $[A, B]$. □

Proposition 3.10. *Given $a > 0$ ($a < \pi$ if $\kappa = 1$) and $\alpha \in (0, \pi)$ there exists an isosceles triangle in M^2_κ with base a and base angles α if and only if $\alpha \in (0, \alpha_{\kappa,a})$, where* $\alpha_{\kappa,a} := \begin{cases} \pi/2 & \text{if } \kappa = 0, \\ \pi & \text{if } \kappa = 1, \\ \arccos\left(\tanh\left(\frac{a}{2}\right)\right) & \text{if } \kappa = -1. \end{cases}$

Proof. We give the proof for $\kappa = -1$. The proof for $\kappa \in \{0, 1\}$ is similar and simpler. From Theorem 3.2 it follows that an isosceles triangle in M^2_{-1} with base angles α exists only if $\alpha \in (0, \pi/2)$. Therefore, we consider

$$a > 0 \quad \text{and} \quad \alpha \in (0, \pi/2). \tag{8}$$

Let $p_0 = (0, 0, 1) \in M^2_{-1}$. The geodesic segment

$$\gamma(t) = \gamma_{p_0,e_1}(t) = (\sinh(t), 0, \cosh(t)),\ t \in [0, a],$$

joins p_0 to $q := (\sinh(a), 0, \cosh(a))$ and has length a. The vector

$$v_1 := (\cos\alpha, \sin\alpha, 0) \in T_{p_0}\left(M^2_{-1}\right)$$

makes an angle α with e_1 in $T_{p_0}\left(M^2_{-1}\right)$. Let

$$n := \gamma'\left(\frac{a}{2}\right),\ \tilde{H} := \{x \in \mathbb{R}^3 \mid\ < x, n >_{-1} = 0\} \ \text{ and } \ H := \tilde{H} \cap M^2_{-1}.$$

Let r_H denote the reflection in M^2_{-1} through H. Let

$$v_2 := d(r_H)_{p_0}(v_1) = r_H(v_1) = (-\cosh a\, \cos\alpha, \sin\alpha, -\sinh a\, \cos\alpha).$$

Clearly, v_2 makes an angle α with $-\gamma'(a)$ in $T_q\left(M^2_{-1}\right)$. Consider the geodesics $\gamma_1 = \gamma_{p_0,v_1}$ and $\gamma_2 = \gamma_{q,v_2}$ of M^2_{-1}. Then,

$$\gamma_1(t) = (\sinh t\, \cos\alpha, \sinh t\, \sin\alpha, \cosh t)$$

and,

$$\gamma_2(t) = (\cosh t\,\sinh a - \sinh t\,\cosh a\,\cos\alpha,\ \sinh t\,\sin\alpha, \cosh t\,\cosh a - \sinh t\,\sinh a\,\cos\alpha).$$

So $\gamma_1(t) = \gamma_2(t)$ for some $t \in \mathbb{R} \setminus \{0\}$ if and only if

$$\left.\begin{array}{l} \sinh t\,\cos\alpha = \cosh t\,\sinh a - \sinh t\,\cosh a\,\cos\alpha \\ \text{and} \\ \cosh t = \cosh t\,\cosh a - \sinh t\,\sinh a\,\cos\alpha. \end{array}\right\}$$

That is,

$$\left.\begin{array}{l} \sinh t\,\cos\alpha\,(1+\cosh a) = \cosh t\,\sinh a \\ \text{and} \\ -\cosh t(1-\cosh a) = \sinh t\,\sinh a\,\cos\alpha. \end{array}\right\} \tag{9}$$

Using (A-9), (A-10) and (A-4) it is easy to see that each of the equations in (9) is equivalent to

$$\cos\alpha = \coth t\ \tanh\left(\frac{a}{2}\right). \tag{10}$$

From (8) and (10) we get $t > 0$. Now, for all $t > 0$,

$$\tanh(a/2)\,\coth t \in (\tanh(a/2), \coth t) \subset (0, \infty)$$

since $\tanh(a/2) \in (0,1)$ and $\coth t \in (1,\infty)\ \forall\ t > 0$. Therefore, from (10) it follows that $\cos\alpha \in (0,1) \cap (\tanh(a/2), \coth t) = (\tanh(a/2), 1)$. Thus an isosceles triangle with base a and base angles α exists if and only if $\alpha < \arccos(\tanh(a/2))$. □

Proposition 3.11. *Given $0 < a < s$ ($< \pi$ if $\kappa = 1$) there exists an isosceles triangle in M_κ^2 with base a and equal sides $s - \dfrac{a}{2}$.*

Proof. Let $f(\kappa, a, s) := \dfrac{T_\kappa(a/2)}{T_\kappa\left(s - \frac{a}{2}\right)}$. Then,

$$\begin{array}{ll} f(0,a,s) \in (0,1) & \text{since } s - a/2 > a/2, \\ f(1,a,s) \in (-1,1) & \text{since } 0 < a < s < \pi \\ \text{and} & \\ f(-1,a,s) > \tanh(a/2) \in (0,1) & \text{since } \coth t > 1\ \forall\ t > 0. \end{array}$$

Now let $\alpha(\kappa, a, s) := \arccos(f(\kappa, a, s))$. Then $\alpha(\kappa, a, s) \in (0, \alpha_{\kappa,a})\ \forall\ \kappa \in \{-1, 0, 1\}$. Therefore, by Proposition 3.10, there exists an isosceles triangle

$T_{\kappa,a,s}$ in M^2_κ with base a and base angles $\alpha_{\kappa,a,s}$. Further, equal sides of $T_{\kappa,a,s}$ are $s - a/2$. □

Proposition 3.12. *Among all triangles in M^2_κ with base a and perimeter $2s_0$ ($s_0 < \pi$ if $\kappa = 1$), the isosceles triangle has maximum area.*

Proof. By triangle inequality it follows that $s_0 > a$. By Proposition 3.11, there exists an isosceles triangle T_0 with base a and equal sides $s_0 - \dfrac{a}{2}$. Let T be any triangle in M^2_κ with sides a, b, c such that $a + b + c = 2s_0$. Let A, A_0 denote the areas of triangles T, T_0 respectively. Then, by (2) we get,

$$\left.\begin{aligned} T_{|\kappa|}\left(\frac{A}{4}\right) &= \sqrt{T_\kappa\left(\frac{s_0}{2}\right) T_\kappa\left(\frac{s_0-a}{2}\right) T_\kappa\left(\frac{s_0-b}{2}\right) T_\kappa\left(\frac{s_0-c}{2}\right)} \text{ and,} \\ T_{|\kappa|}\left(\frac{A_0}{4}\right) &= \sqrt{T_\kappa\left(\frac{s_0}{2}\right) T_\kappa\left(\frac{s_0-a}{2}\right) T_\kappa\left(\frac{a}{4}\right) T_\kappa\left(\frac{a}{4}\right)}. \end{aligned}\right\} \tag{11}$$

We show that $A \le A_0$: Note that $\dfrac{A}{4}, \dfrac{A_0}{4} \in I_\kappa$, where

$$I_\kappa := \begin{cases} (0, \frac{\pi}{4}) & \text{if } \kappa = -1 \\ (0, \infty) & \text{if } \kappa = 0 \\ (0, \frac{\pi}{2}) & \text{if } \kappa = 1 \end{cases}$$

and $T_{|\kappa|}$ is increasing on I_κ. Hence $A \le A_0$ if and only if $T_{|\kappa|}\left(\dfrac{A}{4}\right) \le T_{|\kappa|}\left(\dfrac{A_0}{4}\right)$. Then by (11) it is enough to verify that

$$T_\kappa\left(\frac{s_0-b}{2}\right) T_\kappa\left(\frac{s_0-c}{2}\right) \le T^2_\kappa\left(\frac{a}{4}\right). \tag{12}$$

Case (i) $\underline{\kappa = 0}$:

$$\begin{aligned} \text{LHS of (12)} &= \left(\frac{s_0-b}{2}\right)\left(\frac{s_0-c}{2}\right) = \left(\frac{a+c-b}{4}\right)\left(\frac{a+b-c}{4}\right) \\ &= \frac{a^2-(b-c)^2}{16} \le \frac{a^2}{16} = \left(\frac{a}{4}\right)^2 = T_0^2\left(\frac{a}{4}\right) = \text{RHS of (12)}. \end{aligned}$$

Case (ii) $\underline{\kappa \neq 0}$: Using (A-11) and (A-12) we get, LHS of (12) equals

$$\frac{S_\kappa(\frac{s_0-b}{2})\, S_\kappa(\frac{s_0-c}{2})}{C_\kappa(\frac{s_0-b}{2})\, C_\kappa(\frac{s_0-c}{2})} = -\kappa\, \frac{C_\kappa(\frac{2s_0-b-c}{2}) - C_\kappa(\frac{c-b}{2})}{C_\kappa(\frac{2s_0-b-c}{2}) + C_\kappa(\frac{c-b}{2})} = -\kappa\, \frac{C_\kappa(\frac{a}{2}) - C_\kappa(\frac{c-b}{2})}{C_\kappa(\frac{a}{2}) + C_\kappa(\frac{c-b}{2})}.$$

If $b = c = s_0 - \dfrac{a}{2}$ then

$$\text{RHS of (12)} = T^2_\kappa\left(\frac{a}{4}\right) = -\kappa\,\frac{C_\kappa\left(\frac{a}{2}\right) - 1}{C_\kappa\left(\frac{a}{2}\right) + 1}.$$

Since $C_\kappa(\theta) \in \begin{cases} [-1,1] & \text{if } \kappa = 1 \\ [1,\infty) & \text{if } \kappa = -1 \end{cases}$ we get

$$\text{LHS of (12)} = -\kappa\,\frac{C_\kappa\left(\frac{a}{2}\right) - C_\kappa\left(\frac{c-b}{2}\right)}{C_\kappa\left(\frac{a}{2}\right) + C_\kappa\left(\frac{c-b}{2}\right)} \le -\kappa\,\frac{C_\kappa\left(\frac{a}{2}\right) - 1}{C_\kappa\left(\frac{a}{2}\right) + 1} = \text{RHS of (12)}.$$

□

Theorem 3.13. *The following are equivalent for a polygon $\wp$ in M^2_κ:*
(i) *$\wp$ is convex.*
(ii) *$\wp$ is intersection of finitely many closed half-spaces.*
(iii) *The angle at each vertex of $\wp$ lies in $(0,\pi)$.*

Proof. **(i)** $\Longrightarrow$ **(ii)**: Fix $x_0 \in$ interior of $\wp$. Let n be the number of vertices of $\wp$. Let $S^+_1, \ldots, S^+_n$ be the closed half-spaces containing x_0 corresponding to the boundary geodesic segments $\gamma_1, \ldots, \gamma_n$ of $\partial\wp$ respectively. Then we show that $\wp = S^+_1 \cap \cdots \cap S^+_n$:
$\boldsymbol{\wp \subseteq S^+_1 \cap \cdots \cap S^+_n}$: If not, $\exists\, j \in \{1, \ldots, n\}$ such that $\wp \not\subset S^+_j$. So, $\exists\, y_0 \in$ interior $\wp$ such that $y_0 \notin S^+_j$. We can assume that $x_0 \notin \partial S^+_j$. Then by convexity of $\wp$, the convex hull of $\{\gamma_j,\, x_0,\, y_0\} \subset \wp$, and hence an open set of M^2_κ containing midpoint of γ_j is also contained in $\wp$. This contradicts that $\gamma_j \subseteq \partial\wp$. Thus $\wp \subseteq S^+_1 \cap \cdots \cap S^+_n$.
$\boldsymbol{S^+_1 \cap \cdots \cap S^+_n \subseteq \wp}$: If not, $\exists\, y_0 \in S^+_1 \cap \cdots \cap S^+_n$ such that $y_0 \notin \wp$. Consider $\gamma := [y_0, x_0]$. As $x_0 \in \wp$ and $y_0 \notin \wp$, $\exists$ a point $z \in \partial\wp \cap \gamma$ such that $[z, y_0]$ intersects $\wp$ only at z. Let $i \in \{1, \ldots, n\}$ be such that $z \in \gamma_i$. Then $y_0 \in M^2_\kappa \setminus S^+_i$, which gives a contradiction.
(ii) $\Longrightarrow$ **(iii)**: The polygon $\wp$ being an intersection of finitely many closed half-spaces, is convex. Hence at any vertex of $\wp$, the angle of the polygon is less than π.
(iii) $\Longrightarrow$ **(i)**: By **(iii)**, the polygon $\wp$ is locally convex. As $\wp$ is connected, $\wp$ is then a convex polygon. □

The following result follows by Theorem 3.2.

Proposition 3.14. *Let $\wp$ be a convex polygon with n sides. Let $\theta_1, \dots, \theta_n$ be the angles of $\wp$ at its vertices. Then area$(\wp) = \kappa\ \{(\sum_{i=1}^{n} \theta_i) - (n-2)\pi\}$.*

Lemma 3.15. *The perimeter of any convex n-gon in M_1^2 is strictly less than 2π.*

Proof. Let $\wp$ be a convex n-gon with vertices $P_1, \dots, P_n$ arranged in a cyclic order. Put $P_{n+1} := P_1$. Let a_i be the arc-length of the geodesic segment $[P_i, P_{i+1}]\,\forall\, 1 \le i \le n$. As $\wp$ is a proper polygon, $\{\underline{0} =: (0,0,0),\, P_i,\, P_{i+1}\}$ determine a plane H_i in $\mathbb{E}^3$ for each $i \in \{1, \dots, n\}$. Then $\mathbb{E}^3 \setminus H_i$ has two connected components. We call these components having H_i as common boundary as open half-spaces. Let H_i^+ denote the closed half-space in $\mathbb{E}^3$ having H_i as its boundary such that $\wp \subset H_i^+$. Then $X := \cap_{i=1}^{n} H_i^+$ is a solid cone in $\mathbb{E}^3$ with $\underline{0}$ as its vertex.

The plane H containing points P_1, P_2, P_3 intersects X transversely, and $\wp_1 := X \cap H$ is a convex plane-polygon with n sides. Let $Q_1, \dots, Q_n$ be the vertices of $\wp_1$ which occur in a cyclic order. Consider the 'truncated solid cone' X_1 with vertices $\underline{0}, Q_1, \dots, Q_n$, whose boundary consists of polygon $\wp_1$ and plane-triangles $\{\triangle(\underline{0}, Q_i, Q_{i+1})\}_{1 \le i \le n}$. (Here, $Q_{n+1} := Q_1$ and for $1 \le i \le n$, $\triangle(\underline{0}, Q_i, Q_{i+1})$ denotes plane-triangle determined by vertices $\underline{0}$, Q_i & Q_{i+1}).

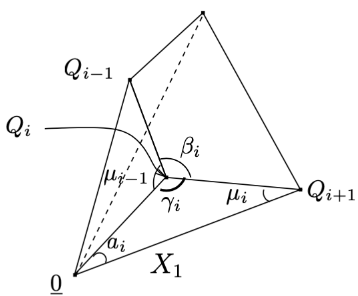

Fig. 2

Clearly, the face-angles of the polyhedra X_1 at the vertex $\underline{0}$ are $a_1, \dots, a_n$. Thus $\sum_{i=1}^{n} a_i$ is the sum of the face angles of X_1 at $\underline{0}$. For each $1 \le i \le n$, let γ_i and μ_i be the angles of the plane-triangle $\triangle(\underline{0}, Q_i, Q_{i+1})$ at the vertices Q_i, Q_{i+1} respectively. Let β_i be the angle of the polygon $\wp_1$ at vertex

$Q_i \, \forall \, 1 \leq i \leq n$. Note that γ_i, μ_i, $\beta_i \in (0, \pi) \, \forall \, i = 1, \ldots, n$. Consider a sphere S with center Q_i having sufficiently small radius $r > 0$. Then $S \cap X_1$ is a triangle in S whose sides are of length $r \, \gamma_i$, $r \, \mu_i$ & $r \, \beta_i$. Thus strict triangle inequality holds and we get $\beta_i < \gamma_i + \mu_{i-1}$, $\forall \, i \in \{1, \ldots, n\}$ $(\mu_0 := \mu_n)$. Therefore,

$$\begin{aligned} n\pi &= \sum_{i=1}^{n} (a_i + \mu_i + \gamma_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} (\mu_{i-1} + \gamma_i) \\ &> \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} a_i + (n-2)\pi. \end{aligned}$$

Hence, perimeter of $\wp = \sum_{i=1}^{n} a_i < 2\pi$. □

Lemma 3.16. ***(Lemma of Cauchy)*** *Let $\wp$ and $\bar{\wp}$ be two convex n-gons in M_κ^2 with respective vertices $\{P_i\}_{i=1,\ldots,n}$ and $\{\bar{P}_i\}_{i=1,\ldots,n}$ occurring in a cyclic order. Let $a_i := d_\kappa(P_i, P_{i+1})$ & $\bar{a}_i := d_\kappa(\bar{P}_i, \bar{P}_{i+1})$ $(1 \leq i \leq n-1)$ denote the lengths of $(n-1)$ sides of $\wp$ and $\bar{\wp}$ respectively. For $2 \leq i \leq n-1$, let α_i (resp. $\bar{\alpha}_i$) denote the angle of $\wp$ (resp. $\bar{\wp}$) at vertex P_i (resp. $\bar{P}_i$) of $\wp$ (resp. $\bar{\wp}$). Let a_n (resp. $\bar{a}_n$) be the length of 'remaining' side of $\wp$ (resp. $\bar{\wp}$). If $a_i = \bar{a}_i$ for all $i = 1, \ldots, n-1$ and $\alpha_i \leq \bar{\alpha}_i$ for all $i = 2, \ldots, n-1$ then $a_n \leq \bar{a}_n$ holds. If in addition, there exists $i \in \{2, \ldots, n-1\}$ with $\alpha_i < \bar{\alpha}_i$ then $a_n < \bar{a}_n$.*

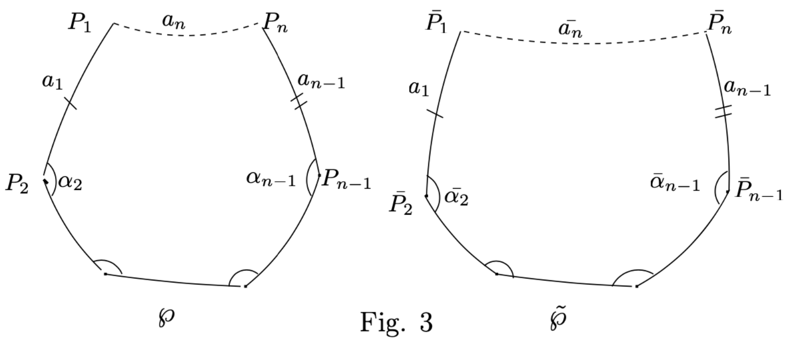

Fig. 3

Proof. By induction on n.

Claim 1: The Lemma of Cauchy is true for $n = 3$.

From the Law of Cosine for a triangle,

$$C_\kappa(a_3) = C_\kappa(a_1) \, C_\kappa(a_2) + \kappa \, S_\kappa(a_1) \, S_\kappa(a_2) \, \cos \alpha_2 \ (\kappa \neq 0),$$

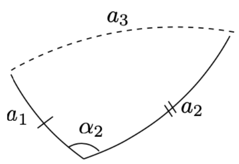

Fig. 4

$$a_3^2 = a_1^2 + a_2^2 - 2\,a_1\,a_2\,\cos\alpha_2 \;(\kappa = 0).$$

Hence it is clear that the side a_3 of a triangle in M_κ^2 with fixed sides a_1 and a_2 is a strictly increasing function of α_2. This proves Claim 1.

Now we assume that the Lemma of Cauchy holds for $n-1\,(n \geq 4)$. Let $\wp$ and $\bar{\wp}$ be two n-gons as in the Lemma of Cauchy.

Claim 2: If $\alpha_i = \bar{\alpha_i}$ for some $i \in \{2, \ldots, n-1\}$, then $a_n \leq \bar{a_n}$. Further, if $\alpha_j < \bar{\alpha_j}$ for some $j \in \{2, \ldots, n-1\} \setminus \{i\}$, then $a_n < \bar{a_n}$.

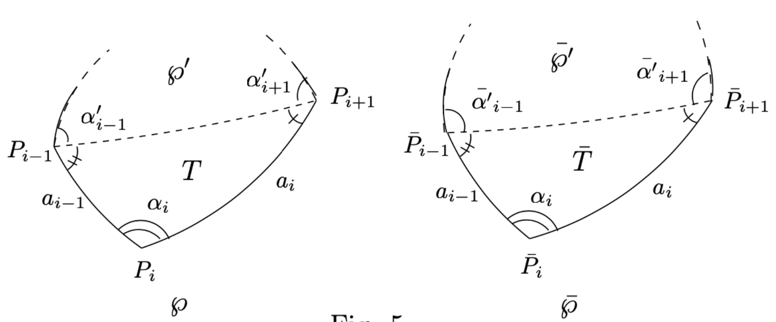

Fig. 5

Let γ_i and $\bar{\gamma_i}$ denote the geodesic segments $[P_{i-1},\ P_{i+1}]$ and $[\bar{P}_{i-1},\ \bar{P}_{i+1}]$ respectively. Since $\wp$ and $\bar{\wp}$ are convex we obtain two convex $(n-1)$-gons $\wp'$ and $\bar{\wp}'$ and two triangles T and $\bar{T}$ as shown in the figure above. Note that T and $\bar{T}$ are congruent triangles. In particular, the angles of T, $\bar{T}$ at P_{i-1} and $\bar{P}_{i-1}$(resp. at P_{i+1} and $\bar{P}_{i+1}$) are equal. This implies that $\alpha'_{i-1} \leq \bar{\alpha}'_{i-1}$ and $\alpha'_{i+1} \leq \bar{\alpha}'_{i+1}$. Here, α'_{i-1}, α'_{i+1} are angles of $\wp'$ at vertices P_{i-1}, P_{i+1} respectively. Similarly $\bar{\alpha}'_{i-1}$, $\bar{\alpha}'_{i+1}$ are defined. Thus $\wp'$ and $\bar{\wp}'$ satisfy the assumption of the Lemma of Cauchy, and Claim 2 follows by induction assumption.

By Claim 2, we can now assume that $\wp$ and $\bar{\wp}$ are two convex n-gons as in the Lemma of Cauchy which further satisfy $\alpha_i < \bar{\alpha}_i \; \forall i \in \{2, \dots, n-1\}$. We show that $a_n < \bar{a_n}$: Increase the angle α_{n-1} of $\wp$ at P_{n-1} until it becomes equal to $\bar{\alpha}_{n-1}$, while maintaining the $(n-1)$ sides constant. This way, we obtain a new polygon $\wp'$ with vertices $P_1, \dots, P_{n-1}, P_n'$, side-lengths $a_1, \dots, a_{n-1}, a_n' := d_\kappa(P_n', P_1)$ and angles at vertices $P_2, \dots, P_{n-2}, P_{n-1}$ equal to $\alpha_2, \dots, \alpha_{n-2}, \bar{\alpha}_{n-1}$ respectively.

Case (i) $\wp'$ is convex:
Join P_1, P_{n-1} by geodesic segment γ (say). Since $\wp'$ is convex, $\gamma \subset \wp'$ and γ divides $\wp'$ into two convex proper polygons. Apply Claim 1 to the two triangles $[P_1, P_{n-1}, P_n]$ and $[P_1, P_{n-1}, P_n']$, whence $a_n = d_\kappa(P_1, P_n) \leq d_\kappa(P_1, P_n') = a_n'$, and $a_n < a_n'$ since $\alpha_{n-1} < \bar{\alpha}_{n-1}$.

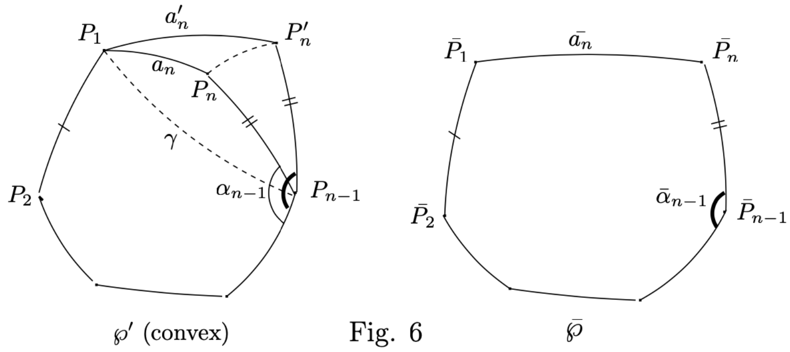

Fig. 6

We then apply the induction assumption and Claim 2 to the n-gons $\wp'$ and $\bar{\wp}$ which have the same angles at P_{n-1} and $\bar{P}_{n-1}$, and obtain

$$a_n' = d_\kappa(P_1, P_n') \leq d_\kappa(\bar{P}_1, \bar{P}_n) = \bar{a_n}.$$

Thus $a_n < a_n' \leq \bar{a_n}$ and we have concluded the proof for case (i).

Case (ii) $\wp'$ is not convex:
In this case, as we increase α_{n-1} by rotating side $[P_{n-1}, P_n]$ around P_{n-1}, there exists a smallest value α_{n-1}' of the angle for which $\wp'$ ceases to be convex. This value lies between α_{n-1} and $\bar{\alpha}_{n-1}$.

Let P_n'' be the point thus obtained. By construction, P_n'' belongs to the

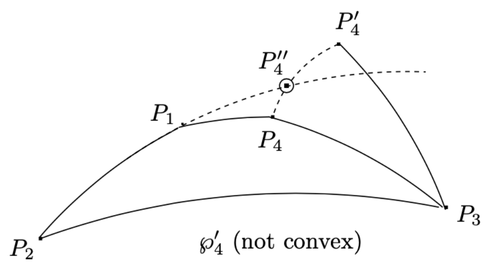

Fig. 7

line determined by P_2 and P_1. We have

$$a_n'' := d_\kappa(P_1, P_n'') = d_\kappa(P_2, P_n'') - d_\kappa(P_1, P_2) = d_\kappa(P_2, P_n'') - a_1 \qquad (13)$$

Applying triangle inequality to the triangle $[\bar{P}_1, \bar{P}_2, \bar{P}_n]$ we get

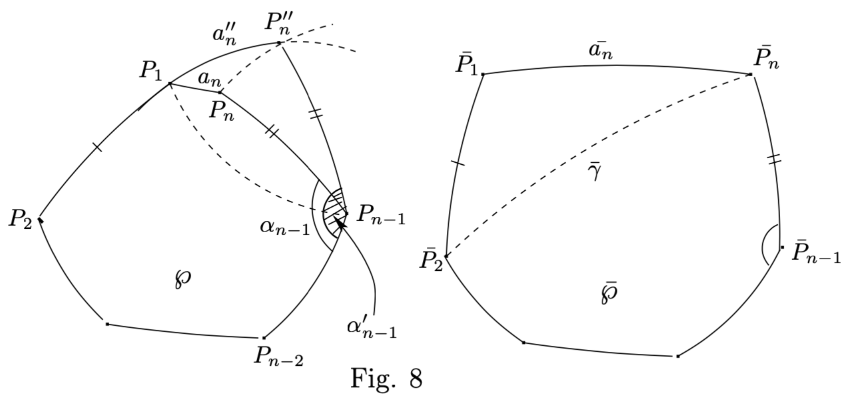

Fig. 8

$$\begin{aligned} \bar{a_n} = d_\kappa(\bar{P}_1, \bar{P}_n) &\geq d_\kappa(\bar{P}_2, \bar{P}_n) - d_\kappa(\bar{P}_1, \bar{P}_2) \\ &= d_\kappa(\bar{P}_2, \bar{P}_n) - d_\kappa(P_1, P_2) \\ &= d_\kappa(\bar{P}_2, \bar{P}_n) - a_1. \qquad (14) \end{aligned}$$

Now we can apply induction assumption to the convex $(n-1)$-gons $[\bar{P}_2, \dots, \bar{P}_n]$ and $[P_2, P_3, \dots, P_{n-1}, P_n'']$ to get

$$d_\kappa(\bar{P}_2, \bar{P}_n) \geq d_\kappa(P_2, P_n''). \qquad (15)$$

Finally, applying Claim 1 to the triangles $[P_1, P_n, P_{n-1}]$ and $[P_1, P_n'', P_{n-1}]$ we get

$$a_n'' > a_n \qquad (16)$$

Thus,

$$\begin{aligned}\bar{a_n} &\geq d_\kappa(\bar{P_2}, \bar{P_n}) - a_1 &&[\text{by } (14)]\\ &\geq d_\kappa(P_2, P''_n) - a_1 &&[\text{by } (15)]\\ &= a''_n &&[\text{by } (13)]\\ &> a_n &&[\text{by } (16)].\end{aligned}$$

This proves the Lemma of Cauchy. □

Lemma 3.17. *Among all convex n-gons in M^2_κ whose all sides but one are given in length – say $a_1, \ldots, a_{n-1}$ – (with $a_1 + \cdots + a_{n-1} < \pi$ if $\kappa = 1$), area maximizer is the convex n-gon whose vertices lie on a circle having its center at the midpoint of the remaining side.*

Proof. When $n = 3$, this result is proved in Proposition 3.9. Here we consider $n \geq 4$. Let $\mathcal{U}$ denote the family of all convex n-gons in M^2_κ whose all sides but one are $a_1, \ldots, a_{n-1}$ (with $a_1 + \cdots + a_{n-1} < \pi$ if $\kappa = 1$). Let $r_0 := a_1 + \cdots + a_{n-1}$. *Existence* Upto congruence all polygons in $\mathcal{U}$ lie inside $B_\kappa(p, r_0)$ where $p \in M^2_\kappa$. Since $\overline{B_\kappa(p, r_0)}$ is compact in (M^2_κ, d_κ) and number of vertices is n for all polygons in $\mathcal{U}$, there exists an 'area maximizer' $\wp$ in $\mathcal{U}$.

Let $A_1, \ldots, A_n\,(A_{n+1} := A_1)$ be the vertices of $\wp$ which occur in a cyclic order and such that $d_\kappa(A_i, A_{i+1}) = a_i \,\forall\, i = 1, \ldots, n-1$. Put $r := d_\kappa(A_1, A_n)/2$ and $O :=$ mid-point of geodesic segment $[A_1, A_n]$. We show that $d_\kappa(O, A_i) = r \;\forall\; i = 1, \ldots, n$:

Suppose $d_\kappa(O, A_i) \neq r$ for some $i \in \{2, \ldots, n-1\}$. Put $a := d_\kappa(A_1, A_i)$ and $b := d_\kappa(A_n, A_i)$. Since d_κ is a metric on M^2_κ we get $a + b \leq a_1 + a_2 + \cdots a_{n-1}$ ($< \pi$ if $\kappa = 1$), and by assumption A_i does not lie on the circle of radius r and center O. By Proposition 3.9, there exists a triangle $[A'_1, A_i, A'_n]$ such that $d_\kappa(A'_1, A_i) = a$, $d_\kappa(A'_n, A_i) = b$ and

$$area([A'_1, A_i, A'_n]) > area([A_1, A_i, A_n]) \tag{17}$$

Further we can assume that the angles of $[A'_1, A_i, A'_n]$ at vertices A'_1, A_i, A'_n are close to the angles of $[A_1, A_i, A_n]$ at vertices A_1, A_i, A_n respectively.

Let T' be the triangle $[A'_1, A_i, A'_n]$. Let S_1 be the closed half-space of M^2_κ containing A'_n and having the *line* containing $[A'_1, A_i]$ as its boundary. Let S_2 denote the other closed half-space. Consider the polygon $\wp'_1 \subset S_2$ with vertices $A'_1, A'_2, \dots, A'_i$ with $A'_i = A_i$, occurring in a cyclic order such that $\wp'_1$ is congruent to the convex polygon $[A_1, A_2, \dots, A_i]$. Let S_3 be the closed half-space of M^2_κ containing A'_1 and having the *line* containing $[A_i, A'_n]$ as its boundary. Let S_4 denote the other closed half-space. Similarly, consider a polygon $\wp'_2 \subset S_4$ with vertices $A'_i(= A_i), A'_{i+1}, \dots, A'_n$ occurring in a cyclic order such that $\wp'_2$ is congruent to the convex polygon $[A'_i, A_{i+1}, \dots, A_n]$. Polygons $\wp'_1$ and $\wp'_2$ do not intersect the interior of T'. Thus we have constructed a polygon $\wp'$, with n vertices A'_1, A'_2, $\dots$, A'_n occurring in a cyclic order and such that $\wp' = \wp'_1 \cup T' \cup \wp'_2$. By (17) and construction of $\wp'$, $area(\wp') > area(\wp)$. Also, since the angles of T' at vertices A'_1, A_i, A'_n are *sufficiently close* to the angles of $[A_1, A_i, A_n]$ at vertices A_1, A_i, A_n respectively, then the angles of $\wp'$ at the vertices A'_1, $A'_i = A_i$, A'_n are strictly less than π. By Theorem 3.13 it follows that $\wp'$ is a *convex polygon*. Thus $\wp' \in \mathcal{U}$ and $area(\wp') > area(\wp)$, which contradicts the fact that $\wp$ is an 'area maximizer' in $\mathcal{U}$. We conclude that $d_\kappa(O, A_i) = r \ \forall\ i \in \{1, \dots, n\}$. $\square$

Lemma 3.18. *Let $\mathcal{C}$ be any piecewise smooth closed curve in M^2_1 whose arc-length is strictly less than 2π. Then $\mathcal{C}$ is contained in an open hemisphere.*

(cf. [34])

Definition 3.19. A *digon* $D_{x,\alpha}$ ($x \in M^2_1$ and $\alpha \in [0, \pi]$) is a closed region of M^2_1 bounded by two half great circles with end points $x, -x$ and forming an angle α at x.

Remark 3.20. The area of the digon

$$D_{p_0,\alpha} = \left\{(\cos\theta\ \cos\phi,\ \cos\theta\ \sin\phi,\ \sin\theta) : \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \phi \in [0, \alpha]\right\}$$

is equal to 2α since $\displaystyle\int_0^\alpha \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta\, d\phi = 2\alpha$. There is an isometry between any two digons with the same angle α. Hence for each $\alpha \in [0, \pi]$, area of digon $D_{x,\alpha}$ is $2\,\alpha\ \forall\ x \in M^2_1$.

Lemma 3.21. *Let $\mathcal{C}$ be a piecewise smooth closed curve in M^2_1 with arc-length 2π. If $\mathcal{C}$ is not a digon then $\mathcal{C}$ is contained in an open hemisphere.*

(cf. [34])

4. REGULAR POLYGONS IN M^2_κ

A polygon in M^2_κ is said to be *equilateral* (resp. *equiangular*) if all its sides have same length (resp. if all its angles are equal). A polygon is said to be *regular* if it is convex, equilateral and equiangular. A regular polygon (proper regular polygon if $\kappa = 1$) of n sides is called a *regular n-gon*.

Construction of regular polygons in M^2_κ: Fix $r > 0$ ($r < \frac{\pi}{2}$ if $\kappa = 1$) and $n \geq 3$. Let $p_0 \in M^2_\kappa$ be as in (1). Let $\mathcal{C}_\kappa(p_0, r)$ denote the circle which is the boundary of the disc $B_\kappa(p_0\,, r)$ contained in M^2_κ. Then $\mathcal{C}_\kappa(p_0, r)$ is nothing but a Euclidean circle in the plane $\{(x_1, x_2, |\kappa|\, C_\kappa(r)) \mid x_1, x_2 \in \mathbb{R}\} \subseteq \mathbb{R}^3$ with center $\underline{c} = C_\kappa(r)\, p_0$ and radius $S_\kappa(r)$.

Let $P_1, \ldots, P_n$ be n points in $\mathcal{C}_\kappa(p_0, r)$ which occur clockwise such that $\measuredangle\{P_i - \underline{c}\,, P_{i+1} - \underline{c}\} = \frac{2\pi}{n}$ $\forall\, i = 1, \ldots, n$ (here, $P_{n+1} := P_1$). Let $\wp_{n,r}$ denote the convex polygon in M^2_κ with $P_1, \ldots, P_n$ as its vertices. By construction, the rotation, $\rho_{\frac{2\pi}{n}}$ about the oriented axis through $\underline{c}$ normal to the plane of $\mathcal{C}_\kappa(p_0, r)$ is a symmetry of $\wp_{n,r}$. Here the axis is oriented by the vector $(0, 0, 1)$. Thus, $\wp_{n,r}$ is an equilateral, equiangular n-gon in M^2_κ. Any two convex polygons constructed as above are congruent to each other for a fixed $n \geq 3$ and fixed $r > 0$ ($r < \frac{\pi}{2}$ if $\kappa = 1$).

Let a be the length of a side of $\wp_{n,r}$. Let Q be the midpoint of $[P_1, P_2]$. Then the triangles $[p_0, Q, P_1]$ is congruent to the triangle $[p_0, Q, P_2]$ and for both these triangles the angle at the vertex Q is $\pi/2$. The Law of Sine (B-3) applied to the triangle $[p_0, Q, P_1]$ gives $\dfrac{S_\kappa(r)}{\sin\left(\frac{\pi}{2}\right)} = \dfrac{S_\kappa\left(\frac{a}{2}\right)}{\sin\left(\frac{\pi}{n}\right)}$. Therefore,

$$a = a(n, r) = 2\, AS_\kappa\left(S_\kappa(r)\ \sin\left(\frac{\pi}{n}\right)\right). \tag{18}$$

Now we compute the angle $\theta = \theta(n, r)$ at vertices of $\wp_{n,r}$. Recall that $n \geq 3$. The Law of Sine (B-3) applied to the triangle $[P_1, P_2, p_0]$ we get

$$\frac{S_\kappa(r)}{\sin\left(\frac{\theta}{2}\right)} = \frac{S_\kappa(a)}{\sin\left(\frac{2\pi}{n}\right)}. \tag{19}$$

$\underline{\kappa = 0}$ From (18) and (19) it follows that

$$\sin\left(\frac{\theta}{2}\right) = \frac{2\,\sin\left(\frac{\pi}{n}\right)\,\cos\left(\frac{\pi}{n}\right)\, S_\kappa(r)}{2\,\sin\left(\frac{\pi}{n}\right)\, S_\kappa(r)} = \cos\left(\frac{\pi}{n}\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right).$$

$$\text{Therefore, } \theta = \theta(n, r) = \left(\frac{n-2}{n}\right)\pi \qquad (\kappa = 0). \tag{20}$$

$\underline{\kappa \neq 0}$ From (18) and (19) we get,

$$\frac{S_\kappa(r)}{\sin\left(\frac{\theta}{2}\right)} = \frac{S_\kappa(a)}{\sin\left(\frac{2\pi}{n}\right)} = \frac{2\, S_\kappa\left(\frac{a}{2}\right)\, C_\kappa\left(\frac{a}{2}\right)}{2\,\sin\left(\frac{\pi}{n}\right)\,\cos\left(\frac{\pi}{n}\right)} = \frac{S_\kappa(r)\,\sin\left(\frac{\pi}{n}\right)\, C_\kappa\left(\frac{a}{2}\right)}{\sin\left(\frac{\pi}{n}\right)\,\cos\left(\frac{\pi}{n}\right)}.$$

Thus

$$\sin\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\pi}{n}\right)}{C_\kappa\left(\frac{a}{2}\right)} \ (\kappa \neq 0). \tag{21}$$

Therefore, by (21) and (18),

$$\begin{aligned}
\cos\left(\frac{\theta}{2}\right) &= \frac{\sqrt{C_\kappa^2\left(\frac{a}{2}\right) - \cos^2\left(\frac{\pi}{n}\right)}}{C_\kappa\left(\frac{a}{2}\right)} = \frac{\sqrt{1-\kappa\, S_\kappa^2\left(\frac{a}{2}\right) - \cos^2\left(\frac{\pi}{n}\right)}}{C_\kappa\left(\dfrac{a}{2}\right)} \\
&= \frac{\sqrt{\sin^2\left(\frac{\pi}{n}\right) - \kappa\, S_\kappa^2(r)\, \sin^2\left(\frac{\pi}{n}\right)}}{C_\kappa\left(\frac{a}{2}\right)} \\
&= \frac{\sqrt{\sin^2\left(\frac{\pi}{n}\right)\left(1-\kappa\, S_\kappa^2(r)\right)}}{C_\kappa\left(\frac{a}{2}\right)} = \frac{\sqrt{\sin^2\left(\frac{\pi}{n}\right)\, C_\kappa^2(r)}}{C_\kappa\left(\frac{a}{2}\right)}.
\end{aligned}$$

Thus

$$\cos\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{\pi}{n}\right)\, C_\kappa(r)}{C_\kappa\left(\frac{a}{2}\right)} \ (\kappa \neq 0). \tag{22}$$

From (21) and (22) we get,

$$\tan\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\pi}{n}\right)}{C_\kappa(r)\, \sin\left(\frac{\pi}{n}\right)} \ (\kappa \neq 0).$$

Therefore

$$\theta = \theta(n,r) = 2\,\arctan\left(\frac{\cot\left(\dfrac{\pi}{n}\right)}{C_\kappa(r)}\right) \ (\kappa \neq 0). \tag{23}$$

Let A denote the area of the regular n-gon $\wp_{n,r}$.
$\underline{\kappa \neq 0}$: By Proposition 3.14, $A = \kappa\ \{n\,\theta - (n-2)\pi\}$. Therefore by (23),

$$A = A(n,r) = \kappa\ \left\{2\,n\,\arctan\left(\frac{\cot\left(\frac{\pi}{n}\right)}{C_\kappa(r)}\right) - (n-2)\pi\right\} \ (\kappa \neq 0). \tag{24}$$

$\underline{\kappa = 0}$: Let T be the triangle $[p_0, P_1, P_2]$. Then $area(T) = \dfrac{1}{2}\, r^2\, \sin\left(\dfrac{2\,\pi}{n}\right)$.
Hence,

$$A = A(n,r) = n\ area(T) = \frac{n\, r^2}{2} \sin\left(\frac{2\,\pi}{n}\right) \ (\kappa = 0). \tag{25}$$

Theorem 4.1. *Any regular n-gon in M_κ^2 is congruent to $\wp_{n,r}$ for a unique $r > 0$ ($r < \frac{\pi}{2}$ if $\kappa = 1$).*

Proof. Let $\wp'$ be any regular n-gon in M_κ^2. As $\wp'$ is a regular n-gon, $n \geq 3$ holds. Let a' be the length of a side of $\wp'$. By Lemma 2.11, $na' < 2\pi$ if

$\kappa = 1$. That is,

$$a' \in \begin{cases} (0, \frac{2\pi}{n}) & \text{if } \kappa = 1 \\ (0, \infty) & \text{if } \kappa \neq 1. \end{cases}$$

Let

$$J_\kappa := \begin{cases} (0, \pi/2) & \text{if } \kappa = 1 \\ (0, \infty) & \text{if } \kappa \neq 1 \end{cases} \quad \text{and} \quad J'_\kappa := \begin{cases} (0, 2\pi/n) & \text{if } \kappa = 1 \\ (0, \infty) & \text{if } \kappa \neq 1. \end{cases}$$

Consider the function $f : J_\kappa \to J'_\kappa$ defined by $f(r) := 2\, AS_\kappa \left(\sin\left(\frac{\pi}{n}\right) S_\kappa(r) \right)$. Then $f : J_\kappa \to J'_\kappa$ is a bijection for a fixed $n \geq 3$. Hence for $a' \in J'_\kappa$, $\exists$ a unique $r \in J_\kappa$ such that $a' = f(r)$. Thus $a' = a(n, r)$ for a unique $r \in J_\kappa$.

Now we prove that $\wp'$ is congruent to $\wp_{n,r}$. Let $P'_1, \ldots, P'_n$ (resp. P_1, …, P_n) be the vertices of $\wp'$ (resp. of $\wp_{n,r}$) which occur in a cyclic order. Let θ' (resp. θ) be the angle of $\wp'$ (resp. of $\wp_{n,r}$) at its vertices. If $\theta' < \theta$ (resp. $> \theta$), then by the *Lemma of Cauchy* (Lemma 3.16), $d_\kappa(P'_1, P'_n) < d_\kappa(P_1, P_n)$ (resp. $> d_\kappa(P_1, P_n)$) which contradicts that $a' = a(n, r)$. So, $\theta' = \theta$. Applying the *Lemma of Cauchy* again to the convex polygons $[P_1, P_2 \ldots, P_j]$ & $[P'_1, P'_2 \ldots, P'_j]$, we get $d_\kappa(P'_1, P'_j) = d_\kappa(P_1, P_j) \forall\ j = 2, \ldots, n$. Similarly, $d_\kappa(P'_i, P'_j) = d_\kappa(P_i, P_j) \forall\ i, j \in \{1, \ldots, n\}$. By Proposition 2.4, there exists an isometry φ of M_κ^2 such that $\varphi(\wp') = \wp_{n,r}$. $\square$

Proposition 4.2. *Let $\wp$ be a regular n-gon in M_κ^2 having side a, angle θ and area A. Then $\exists$ a unique $r > 0$ ($r < \frac{\pi}{2}$ if $\kappa = 1$) such that $\wp$ is inscribed in a circle of radius r. Further, equations*

(i) $$r = r(n, a) = AS_\kappa \left(\frac{S_\kappa(a/2)}{\sin(\pi/n)} \right), \tag{26}$$

(ii) $$r = r(n, \theta) = AC_\kappa \left(\frac{\cot(\pi/n)}{\tan(\theta/2)} \right) \quad (\kappa \neq 0), \tag{27}$$

(iii) $$r = r(n, A) = \begin{cases} AC_\kappa \left(\cot\left(\frac{\pi}{n}\right) \tan\left(\frac{2\pi - \kappa A}{2n} \right) \right) & \text{if } \kappa \neq 0 \quad (28) \\ \sqrt{\dfrac{2A}{n \sin\left(\frac{2\pi}{n}\right)}} & \text{if } \kappa = 0 \quad (29) \end{cases}$$

hold, and any regular n-gon in M_κ^2 ($\kappa \neq 0$) is determined (uniquely up to congruence) by any one of three:

$$a \in \begin{cases} (0, \pi) & \text{if } \kappa = 1 \\ (0, \infty) & \text{if } \kappa = -1 \end{cases}, \quad \theta \in (0, \pi), \ A \in \begin{cases} (0, 2\pi) & \text{if } \kappa = 1 \\ (0, (n-2)\pi) & \text{if } \kappa = -1 \end{cases}.$$

Further, any regular n-gon in M_0^2 is determined (uniquely up to congruence) by any one of two: $a, A \in (0,\infty)$.

Proof. By the Theorem 4.1, there exists a unique $r > 0$ ($r < \frac{\pi}{2}$ if $\kappa = 1$) such that $\wp$ is congruent to $\wp_{n,r}$. Now equation (26), (27), (28) and (29) easily follows from (18), (23), (24) and (25) respectively.

Finally, it can be verified that the functions in **(i)**, **(ii)**, **(iii)** are strictly monotone functions and hence $\wp$ is determined up to congruence by any one of the entities θ (if $\kappa \neq 0$), a and A. ☐

Remark 4.3. Fix $n \in \mathbb{N}$. When $\kappa = 0$ the angle $\theta \in (0,\pi)$ is not enough to determine the regular n-gon. But for $\kappa \neq 0$ the regular n-gons in M_κ^2 with angle θ are congruent.

Corollary 4.4. *Let $(\tilde{\wp}_k)_{k\in\mathbb{N}}$ be a sequence of regular polygons in M_κ^2 (proper regular polygons if $\kappa = 1$) such that $\tilde{\wp}_k$ has k vertices $\forall\ k$, and $A_k :=$ $(area(\tilde{\wp}_k)) \longrightarrow A'$ as $k \longrightarrow \infty$. For each $k \in \mathbb{N}$, let r_k be the radius of the circle in which $\tilde{\wp}_k$ is inscribed. Then*

(i) $\displaystyle\lim_{k\longrightarrow\infty} r_k = AS_\kappa\left(\sqrt{A'\,(4\pi - \kappa\, A')}/(2\pi)\right)$

(ii) $\displaystyle\lim_{k\longrightarrow\infty} (\,perimeter(\tilde{\wp}_k)) = \sqrt{A'\,(4\pi - \kappa\, A')}.$

Proof. **(i)** $\underline{\kappa = 0}$: By (29), $r_k^2 = r(k, A_k)^2 = \dfrac{2\,A_k}{k\,\sin\left(\frac{2\pi}{k}\right)}$. Therefore,

$$\lim_{k\longrightarrow\infty} r_k^2 = \lim_{k\longrightarrow\infty} \frac{2\,A_k}{2\,\pi}\frac{1}{\frac{\sin\left(\frac{2\pi}{k}\right)}{\left(\frac{2\pi}{k}\right)}} = \frac{A'}{\pi}.$$

Thus

$$\lim_{k\longrightarrow\infty} r_k = \sqrt{\frac{A'}{\pi}}.$$

$\underline{\kappa \neq 0}$: By (24),

$$\frac{\cot\left(\frac{\pi}{k}\right)}{C_\kappa(r_k)} = \tan\left(\frac{A_k + \kappa\,(k-2)\,\pi}{2\,\kappa\,k}\right) = \tan\left(\frac{\kappa\,A_k - 2\pi}{2\,k} + \frac{\pi}{2}\right) = \cot\left(\frac{2\pi - \kappa\,A_k}{2\,k}\right).$$

Hence,

$$C_\kappa(r_k) = \cot\left(\frac{\pi}{k}\right)\,\tan\left(\frac{2\pi - \kappa\,A_k}{2\,k}\right).$$

Therefore,

$$\begin{aligned}\kappa\, S_\kappa^2(r_k) &= 1 - C_\kappa^2(r_k) = \frac{\sin^2\left(\frac{\pi}{k}\right)\,\cos^2\left(\frac{2\pi-\kappa\,A_k}{2\,k}\right) - \cos^2\left(\frac{\pi}{k}\right)\,\sin^2\left(\frac{2\pi-\kappa\,A_k}{2\,k}\right)}{\sin^2\left(\frac{\pi}{k}\right)\,\cos^2\left(\frac{2\pi-\kappa\,A_k}{2\,k}\right)} \\ &= \frac{\sin\left(\frac{4\,\pi-\kappa\,A_k}{2\,k}\right)\,\sin\left(\frac{\kappa\,A_k}{2\,k}\right)}{\sin^2\left(\frac{\pi}{k}\right)\,\cos^2\left(\frac{2\pi-\kappa\,A_k}{2\,k}\right)}.\end{aligned}$$

That is,

$$\begin{aligned}S_\kappa^2(r_k) &= \frac{\kappa\,\sin\left(\frac{\kappa\,A_k}{2\,k}\right)\,\sin\left(\frac{4\,\pi-\kappa\,A_k}{2\,k}\right)}{\sin^2\left(\frac{\pi}{k}\right)\,\cos^2\left(\frac{2\pi-\kappa\,A_k}{2\,k}\right)} \\ &= \frac{\sin\left(\frac{A_k}{2\,k}\right)\,\sin\left(\frac{4\,\pi-\kappa\,A_k}{2\,k}\right)}{\sin^2\left(\frac{\pi}{k}\right)\,\cos^2\left(\frac{2\pi-\kappa\,A_k}{2\,k}\right)}.\end{aligned}$$

So,

$$\lim_{k\longrightarrow\infty} S_\kappa^2(r_k) = \frac{(4\pi - \kappa\,A')\;A'}{4\pi^2}.$$

Hence,

$$\lim_{k\longrightarrow\infty} r_k = AS_\kappa\left(\frac{\sqrt{(4\pi-\kappa\,A')\;A'}}{2\,\pi}\right).$$

(ii) Put $r_0 = AS_\kappa\left(\sqrt{A'\,(4\pi-\kappa\,A')}/(2\,\pi)\right)$. Now, each $\tilde{\wp_k}$ is a regular k-gon inscribed in a circle of radius r_k in M_κ^2, and $(r_k) \longrightarrow r_0$ as $k \longrightarrow \infty$ by **(i)**. Hence, $(perimeter(\tilde{\wp_k}))$ converges to the perimeter of the circle of radius r_0 in M_κ^2. Therefore, $\lim\limits_{k\longrightarrow\infty}(perimeter(\tilde{\wp_k})) = 2\pi\,S_\kappa(r_0) = \sqrt{A'\,(4\pi-\kappa A')}$. □

5. ISOPERIMETRIC PROBLEM FOR POLYGONS IN M_κ^2

Proof of Theorem 1.1: Fix $n \geq 3$ in $\mathbb{N}$ & $A \in \begin{cases}(0,\infty) & \text{if } \kappa = 0,\\ (0,2\pi) & \text{if } \kappa = 1,\\ (0,(n-2)\,\pi) & \text{if } \kappa = -1.\end{cases}$

Let $\mathcal{F}$ be the family of all polygons with n vertices in M_κ^2 having area at least A. By Proposition 4.2, there exists a regular n-gon $\wp_{n,r}$ of area equal to A. So, $\mathcal{F}$ is a nonempty family. Define $L = glb\,\{perimeter(\wp) : \wp \in \mathcal{F}\}$. By Lemma 2.11, $perimeter(\wp_{n,r}) < 2\pi$ if $\kappa = 1$. Hence, $L < 2\pi$ if $\kappa = 1$. Let $(\wp_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ such that $(perimeter(\wp_k)) \searrow L$ as $k \longrightarrow \infty$

and $perimeter(\wp_k) < 2\pi \ \forall\ k$ if $\kappa = 1$. We assume p_0 as in (1) is a vertex of $\wp_k \ \forall\ k \in \mathbb{N}$. By Lemma 2.14, we can assume that if $\kappa = 1$ then each $\wp_k$ is contained in the open hemisphere $B_1(p_0, \pi/2)$. Let $X_k^{(1)}$, $X_k^{(2)}$, ..., $X_k^{(n)}$ be the vertices of $\wp_k$ with $X_k^{(1)} = p_0$ occurring in a cyclic order (determined by 'boundary orientation' of $\partial\wp_k$) for all $k \in \mathbb{N}$. Then without loss of generality $\wp_k \in B_\kappa(p_0, L+1) \ \forall\ k$ when $\kappa \neq 1$. As M_1^2 is a compact manifold and $\overline{B_\kappa(p_0, L+1)}$ is compact in M_κ^2 for $\kappa \neq 1$, each sequence $\left(X_k^{(j)}\right)_{k\in\mathbb{N}}$ admits a converging subsequence $\forall\ j = 1, \ldots, n$. Thus, without loss of generality, we can assume that $\left(X_k^{(j)}\right)_{k\in\mathbb{N}}$ converges to some Y_j in M_κ^2, $\forall\ j = 1, \ldots, n$. Clearly, $Y_1 = p_0$. Let $Y_{n+1} := Y_1$. Then $\cup_{i=1}^n [Y_i, Y_{i+1}]$ is a simple closed curve in M_κ^2.

When $\kappa = 1$, $L = \sum_{i=1}^n d_1(Y_i, Y_{i+1}) < 2\,\pi$. and hence by Lemma 2.14, there exists a polygon $\wp_0$ contained in an open hemisphere of M_1^2 having $Y_1, Y_2, \ldots, Y_n$ as its vertices occurring in a cyclic order. In particular, $\wp_0$ is a proper polygon when $\kappa = 1$. When $\kappa \neq 1$, let $\wp_0$ be the polygon in M_κ^2 with $\cup_{i=1}^n [Y_i, Y_{i+1}]$ as its boundary. Then $perimeter(\wp_0) = L$ and $area(\wp_0) \geq A$. It remains to show that $\wp_0$ is a regular n-gon.

***$\wp_0$ is convex*:** If not, by Theorem 3.13, $\exists\, j \in \{1, \ldots, n\}$ such that $\wp_0$ is not locally convex at the vertex Y_j. Put $Y_0 := Y_n$ and $Y_{n+1} := Y_1$. Then we can choose a point Y_j' on the side $[Y_{j-1}, Y_j]$ such that the triangle $[Y_j', Y_j, Y_{j+1}]$ does not intersect interior of $\wp_0$. Now consider polygon $\wp$ in M_κ^2 having $Y_1, Y_2, \ldots, Y_{j-1}, Y_j', Y_{j+1}, \ldots, Y_n$ as its vertices occurring in a cyclic order. Then $perimeter(\wp) < perimeter(\wp_0)$ and $area(\wp) > area(\wp_0) \geq A$. This contradicts the fact that $\wp_0$ is a perimeter minimizer in $\mathcal{F}$. So, $\wp_0$ is a convex n-gon.

$area(\wp_0) = A$: Suppose $area(\wp_0) = A + \delta$ with $\delta > 0$. Choose a point $Y_2' \in [Y_2, Y_3]$ such that $Y_2' \notin \{Y_2, Y_3\}$ and area of the triangle $[Y_1, Y_2, Y_2']$ is less than δ. As $\wp_0$ is convex the triangle $[Y_1, Y_2, Y_2']$ is contained in $\wp_0$. Then the polygon with vertices $Y_1, Y_2', Y_3, \ldots, Y_n$ occurring in a cyclic order has area greater than A and perimeter less than that of $\wp_0$. This is not possible. So, $area(\wp_0) = A$.

***$\wp_0$ is equilateral*:** If $\wp_0$ is not equilateral, then there exists two successive sides of $\wp_0$ which are of unequal lengths. Suppose $d_\kappa(Y_1, Y_2) =: b \neq c := d_\kappa(Y_2, Y_3)$. Join Y_1 and Y_3 by the geodesic segment $[Y_1, Y_3]$. The triangle $[Y_1, Y_2, Y_3]$ is contained in $\wp_0$. The geodesic segment $[Y_1, Y_3]$ as above divides $\wp_0$ in two polygons, namely, triangle $[Y_1, Y_2, Y_3]$ and the $(n-2)$-gon with vertices $Y_1, Y_3, Y_4, \ldots, Y_n$ occurring in a cyclic order. We call this $(n-2)$-gon as $\wp$. By Proposition 3.12, $\exists$ an isosceles triangle $[Y_1, Y_2', Y_3]$ in M_κ^2 such that Y_2', Y_2 lie on the same half-space whose boundary contains $[Y_1, Y_3]$, $d_\kappa(Y_1, Y_2') = (b+c)/2 = d_\kappa(Y_2', Y_3)$ and $area([Y_1, Y_2', Y_3]) > area([Y_1, Y_2, Y_3])$. Then the polygon $\wp \cup [Y_1, Y_2', Y_3]$ is a perimeter minimizer in $\mathcal{F}$ with area strictly greater than A. This is not possible as seen in the previous step. Thus $\wp_0$ is an equilateral n-gon.

Let 'a' denote the side length of the convex equilateral n-gon $\wp_0$.

***$\wp_0$ is equiangular*:** We prove this by considering n even, n odd cases separately.

n is even: Let $n = 2k$. By Lemma 2.11, $ka = \frac{n}{2} a < \pi$ if $\kappa = 1$. Join Y_1 to Y_{1+k} by the geodesic segment $[Y_1, Y_{1+k}]$ contained in $\wp_0$. Let O be the mid-point of $[Y_1, Y_{1+k}]$. Enough to show that $d_\kappa(O, Y_i) = r := d_\kappa(O, Y_1)\ \forall\, i = 2, \ldots, n$.

The segment $[Y_1, Y_{1+k}]$ divides $\wp_0$ into two convex polygons $\wp_1$, $\wp_2$ with $k+1$ vertices. Let ρ be the reflection of M_κ^2 through the *line* containing $[Y_1, Y_{1+k}]$. If $area(\wp_1) > area(\wp_2)$, then $\wp_1 \cup \rho(\wp_1)$ gives a polygon of perimeter L and area greater than A. This is not possible. So, $area(\wp_1) = area(\wp_2)$.

Consider the family of all the convex polygons with $(k+1)$ vertices in M_κ^2 whose all sides but one are of equal length a. Let $\wp_1' = [Y_1', \ldots, Y_{k+1}']$ be the area maximizer in this family with $[Y_1', Y_{k+1}']$ being the 'remaining side'. Then $area(\wp_1') \geq area(\wp_1)$. If $area(\wp_1') > area(\wp_1)$, then we can produce a polygon of perimeter $L = 2ka$ and area greater than A by reflecting $\wp_1'$ through the side $[Y_1', Y_{k+1}']$. Hence $area(\wp_1') = area(\wp_1)$. Now, by lemma 2.13 it follows that $d_\kappa(O, Y_i) = r\ \forall\, i = 1, \ldots, k+1$. Similarly one can prove that $d_\kappa(O, Y_i) = r\ \forall\, i = k+1, \ldots, 2k$.

n is odd: Suppose $\wp_0$ is not equiangular. Let $\wp_R^n$ be a regular n-gon of side a in M_κ^2. By Proposition 3.10, there exists an isosceles triangle T in M_κ^2 having base a and very small angles α at the base. Let $\wp_0^{2n}$ be the polygon with $2n$ sides obtained by 'pasting' triangle congruent to T on each side of $\wp_0$ so that $area(\wp_0^{2n}) = area(\wp_0) + n\ area(T)$. This is possible since $\wp_0$ is a convex polygon. As $\wp_0$ is not equiangular, $\wp_0^{2n}$ is not equiangular. Similarly construct a regular $2n$-gon $\wp_R^{2n}$ by 'pasting' triangle congruent to T on each side of $\wp_R^n$. By the 'n-even' case, as perimeters of $\wp_0^{2n}$ and $\wp_R^{2n}$ are equal, $area(\wp_0^{2n}) < area(\wp_R^{2n})$. Therefore, $area(\wp_0) + n\ area(T) < area(\wp_R^n) + n\ area(T)$. So, $A = area(\wp_0) < area(\wp_R^n)$. Also, $perimeter(\wp_R^n) = n\,a = perimeter(\wp_0)$. Thus $\wp_R^n \in \mathcal{F}$ and $\wp_R^n$ is a 'perimeter minimizer' in $\mathcal{F}$. We have a contradiction as any 'perimeter minimizer' in $\mathcal{F}$ has area A. We conclude that $\wp_0$ is equiangular. □

6. THE ISOPERIMETRIC PROBLEM IN M_κ^2

Notations: For a piecewise smooth simple closed curve γ in M_κ^2 let $\ell(\gamma)$ denote the arc-length of γ. For $\kappa = 1$ if such a curve γ lies in a hemisphere S^+ then γ encloses a domain D_γ contained in S^+. If $\kappa \neq 1$ then such a curve γ always encloses a unique relatively compact domain D_γ contained in M_κ^2. We denote $area(D_\gamma)$ by $A(\gamma)$.

Proof of Theorem 1.2:

Case (i) $\kappa = 1$ and $A = 2\pi$: *There exists a unique perimeter minimizer among all piecewise smooth simple closed curves in M_1^2 enclosing area 2π, and it is a great circle* :

Let $\mathcal{J}$ be the family of all piecewise smooth simple closed curves in M_1^2 enclosing area 2π. Let $S_+^2 := \{(x, y, z) \in S^2 \mid z \geq 0\}$. Since $\partial S_+^2 \in \mathcal{J}$, $\mathcal{J} \neq \emptyset$.

If $\exists\ \mathcal{C} \in \mathcal{J}$ with $\ell(\mathcal{C}) < 2\pi$ then by Lemma 2.14, $\mathcal{C}$ is contained in an open hemisphere. This contradicts the fact that $\mathcal{C}$ encloses area 2π. Hence,

$$\ell(\mathcal{C}) \geq 2\pi\ \forall\, \mathcal{C} \in \mathcal{J}. \tag{30}$$

$$\text{Define } L := \inf\{\ell(\mathcal{C}) \mid \mathcal{C} \in \mathcal{J}\}.$$

Since $\partial S_+^2 \in \mathcal{J}$,

$$L \leq \ell(\partial S_+^2) = 2\pi. \tag{31}$$

Thus from (30) and (31), we get $L = 2\pi$ and ∂S_+^2 is a perimeter minimizer over $\mathcal{J}$.

Let $\mathcal{C}_0$ be a perimeter minimizer over $\mathcal{J}$. That is, $\ell(\mathcal{C}_0) = L = 2\pi$ and $A(\mathcal{C}_0) :=$ area enclosed by $\mathcal{C}_0 = 2\pi$. If $\mathcal{C}_0$ is not boundary of a digon then by Lemma 2.15, $\mathcal{C}_0$ is contained in an open hemisphere, a contradiction again. Hence $\mathcal{C}_0 = \partial D_{x,\alpha}$ for some $x \in M_1^2$ and $\alpha \in [0, \pi]$. Therefore, $2\pi = A(\mathcal{C}_0) = area(D_{x,\alpha}) = 2\alpha$. This implies that $\alpha = \pi$. Thus $\mathcal{C}_0 = \partial D_{x,\pi}$, that is a great circle.

Case (ii) $\kappa \in \{-1, 0, 1\}$ and $A < 2\pi$ if $\kappa = 1$:

Let $p_0 \in M_\kappa^2$ be as in (1). For $r_0 > 0$ ($r_0 < \pi/2$ if $\kappa = 1$), the circle

$\mathcal{C}_{r_0} := \partial B_\kappa(p_0, r_0)$ encloses a domain of area $4\pi S_\kappa^2\left(\frac{r_0}{2}\right)$ ($< 2\pi$ if $\kappa = 1$) with perimeter $2\pi S_\kappa(r_0)$ ($< 2\pi$ if $\kappa = 1$). So, for $\kappa = 1$ we need to consider piecewise smooth simple closed curves of lengths strictly less than 2π only. By Lemma 2.14, any such curve lies in a hemisphere.

Step 1. (Existence) *Among all piecewise smooth simple closed curves in* M_κ^2 *enclosing area* A, *a circle of radius* $AS_\kappa\left(\sqrt{A\,(4\pi - \kappa\,A)}/(2\pi)\right)$ *in* M_κ^2 *has least perimeter*:

Let $\mathcal{G}$ denote the family of all piecewise smooth simple closed curves in M_κ^2 (in $S_+^2 := \left\{(x,y,z) \in S^2 \,|\, z \geq 0\right\}$ if $\kappa = 1$) enclosing area at least A. Let $\mathcal{C} \in \mathcal{G}$ be arbitrary. If $Y_1, Y_2, \ldots, Y_n$ are points on a curve $\mathcal{C} \in \mathcal{G}$ which appear in a cyclic order [with $d_\kappa(Y_i, Y_{i+1}) < \pi\ \forall\ i = 1, \ldots, n$ if $\kappa = 1$ $(Y_{n+1} := Y_1)$], then they determine a polygon with vertices $Y_1, Y_2, \ldots, Y_n$ (which is contained in S_+^2 if $\kappa = 1$). Define $L := glb\,\{\ell(\mathcal{C}) \mid \mathcal{C} \in \mathcal{G}\}$. Let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ such that $\ell(\mathcal{C}_n) \searrow L$ as $n \longrightarrow \infty$. Let $p_0 \in M_\kappa^2$ be as in (1). We may assume that $p_0 \in \mathcal{C}_n$ and that $l(\mathcal{C}_n) \leq L+1,\ \forall\, n \in \mathbb{N}$. Hence $\mathcal{C}_n \subset B_\kappa(p_0, L+1),\ \forall\ n \in \mathbb{N}$. Then $A(\mathcal{C}_n) \leq A\left(B_\kappa(p_0, L+1)\right) = 4\pi S_\kappa^2\left(\frac{L+1}{2}\right)\ \forall\, n \in \mathbb{N}$. Therefore, we may assume, after taking a subsequence of $(\mathcal{C}_n)_{n\in\mathbb{N}}$ if necessary, that

$$\ell(\mathcal{C}_n) \searrow L \ \text{ and } \ \left(A(\mathcal{C}_n)\right)_{n\in\mathbb{N}} \longrightarrow: A' \geq A. \tag{32}$$

As each $\mathcal{C}_n$ is a piecewise smooth simple closed curve, we can approximate $\mathcal{C}_n$ by the boundary of a polygon $\wp_{k(n)}$ in M_κ^2 with $k(n)$ sides: i.e., vertices of $\wp_{k(n)}$ lie on $\mathcal{C}_n$, $0 < \ell(\mathcal{C}_n) - \ell(\partial\wp_{k(n)}) < \frac{1}{n}$ and $|area(\wp_{k(n)}) - A(\mathcal{C}_n)| < \frac{1}{n}\ \ \forall\, n \in \mathbb{N}$. (Here $\partial\wp$ denotes the boundary of $\wp$). Then it follows that

$$\lim_{n\longrightarrow\infty} \ell(\partial\wp_{k(n)}) = L \ \text{ and } \ \lim_{n\longrightarrow\infty} area(\wp_{k(n)}) = A'.$$

Put $A_n := area(\wp_{k(n)})\ \forall\ n \in \mathbb{N}$. By Proposition 4.2, for each $n \in \mathbb{N}$ there exists regular $k(n)$-gon $\tilde{\wp}_{k(n)}$ in M_κ^2 of area A_n. By Theorem 1.1, $\ell(\partial\tilde{\wp}_{k(n)}) \leq \ell(\wp_{k(n)})\ \forall\, n \in \mathbb{N}$. Let $r_{k(n)}$ be the radius of the circle in M_κ^2 in which $\tilde{\wp}_{k(n)}$ is inscribed. By Corollary 3.3,

$$\lim_{n\longrightarrow\infty} r_{k(n)} = AS_\kappa\left(\sqrt{A'\,(4\pi - \kappa\,A')}/(2\pi)\right) =: r_0.$$

Thus,

$$A' = 4\pi S_\kappa^2\left(\frac{r_0}{2}\right). \tag{33}$$

Again by Corollary 3.3, $\lim_{n\longrightarrow\infty} \ell(\partial\tilde{\wp}_{k(n)}) = \sqrt{A'\,(4\pi - \kappa\, A')} = 2\pi S_\kappa(r_0)$. Note that $\ell(\partial\tilde{\wp}_{k(n)}) \leq \ell(\partial\wp_{k(n)}) \leq \ell(\mathcal{C}_n)\ \forall\, n \in \mathbb{N}$. Therefore,

$$L = \lim_{n\longrightarrow\infty} \ell(\mathcal{C}_n) \geq \lim_{n\longrightarrow\infty} \ell(\partial\tilde{\wp}_{k(n)}) = 2\pi S_\kappa(r_0). \tag{34}$$

Let $\mathcal{C}_{r_0}$ denote the circle in M^2_κ (in S^2_+ if $\kappa = 1$) of radius r_0. Then $A(\mathcal{C}_{r_0}) = 4\pi S^2_\kappa\left(\frac{r_0}{2}\right) = A'$ (by (33)). So, by (32), $\mathcal{C}_{r_0} \in \mathcal{G}$, and by the definition of L,

$$L \leq \ell(\mathcal{C}_{r_0}) = 2\pi S_\kappa(r_0). \tag{35}$$

By (34) & (35), $L = 2\pi S_\kappa(r_0) = \ell(\mathcal{C}_{r_0})$. Hence $\mathcal{C}_{r_0}$ is a perimeter minimizer in $\mathcal{G}$.

Finally we show that $A(\mathcal{C}_{r_0}) = A$: If $A(\mathcal{C}_{r_0}) \neq A$ then by (32) & (33), $A(\mathcal{C}_{r_0}) = 4\pi S^2_\kappa\left(\frac{r_0}{2}\right) = A' > A$. Then we can replace a small portion of circle $\mathcal{C}_{r_0}$ by a geodesic arc and produce a curve $\tilde{\mathcal{C}}$ in M^2_κ (in S^2_+ if $\kappa = 1$) with $\ell(\tilde{\mathcal{C}}) < \ell(\mathcal{C}_{r_0}) = L$ and $A < A(\tilde{\mathcal{C}}) < A(\mathcal{C}_{r_0})$. Then $\tilde{\mathcal{C}} \in \mathcal{G}$ with $\ell(\tilde{\mathcal{C}}) < L$. This is not possible. Hence $A(\mathcal{C}_{r_0}) = A$.

Step 2. (Uniqueness) *Among all piecewise smooth simple closed curves in M^2_κ enclosing area A, any perimeter minimizer is a circle in M^2_κ of radius $AS_\kappa\left(\sqrt{A\,(4\pi - \kappa\, A)}/(2\pi)\right)$*:

Consider the family $\mathcal{G}$ of all piecewise smooth simple closed curves in M^2_κ (in S^2_+ if $\kappa = 1$) enclosing area at least A and of perimeter strictly less than 2π for $\kappa = 1$. Put $L := glb\,\{\ell(\mathcal{C}) \mid \mathcal{C} \in \mathcal{G}\}$. In Step 1 above, we have proved the existence of curve in $\mathcal{G}$ which is a perimeter minimizer. Let $\mathcal{C}_0 \in \mathcal{G}$ be any perimeter minimizer. Then $\ell(\mathcal{C}_0) = L$ ($< 2\pi$ if $\kappa = 1$). Let D_0 be the domain in M^2_κ (in S^2_+ if $\kappa = 1$) enclosed by $\mathcal{C}_0$. By the arguments similar to those made in the proof of Theorem 1.1, we can show that D_0 is convex and $area(D_0) = A$.

Fix a point P on $\mathcal{C}_0$. Let Q be the point on $\mathcal{C}_0$ which divides $\mathcal{C}_0$ into two arcs $\mathcal{C}^+_0$, $\mathcal{C}^-_0$ of equal length. As $\ell(\mathcal{C}_0) < 2\pi$ in M^2_1 , $Q \neq -P$ if $\kappa = 1$. Let $[P, Q]$ denote the geodesic segment joining P & Q in D_0. This segment divides D_0 into two regions D^+_0 & D^-_0. If $area(D^+_0) < area(D^-_0)$, then consider $\tilde{D}_0 := D^-_0 \cup \rho(D^-_0)$ where ρ is the reflection in M^2_κ through the *line* containing $[P, Q]$. Then boundary $\tilde{\mathcal{C}}_0$ of $\tilde{D}_0$ is a perimeter minimizer in $\mathcal{G}$ and $area(\tilde{D}_0) > area(D_0)$. This is not possible. Hence, $[P, Q]$ divides D_0 into two regions of equal area.

Let O be the mid-point of $[P, Q]$ and $r_0 := d_\kappa(P, Q)/2$. We show that $d_\kappa(O, M) = r_0\ \forall\ M \in \mathcal{C}_0$: Suppose $\exists\, M \in \mathcal{C}_0$ such that $d_\kappa(O, M) \neq$

r_0. Let D_0^+ be the region containing M with $\mathcal{C}_0^+ \cup [P,Q]$ as its boundary. As D_0 is convex, the triangle $[P,M,Q]$ of M^2_κ is contained in D_0^+. Now, $d_\kappa(P,M) + d_\kappa(M,Q) \leq \ell(\mathcal{C}_0^+) = L/2$ ($< \pi$ if $\kappa = 1$) and M does not lie on the circle in M^2_κ of radius r_0 and center O.

By the arguments similar to those made in the proof of Lemma 2.13, we can construct a domain $\widetilde{D_0^+}$ in M^2_κ (in S^2_+ if $\kappa = 1$) of area strictly bigger than $area(D_0^+)$ whose boundary consists of a curve $\widetilde{\mathcal{C}_0^+}$ which is congruent to $\mathcal{C}_0^+$ and a geodesic segment $[P', Q']$ (P', Q' are the endpoints of $\widetilde{\mathcal{C}_0^+}$). Reflecting $\widetilde{D_0^+}$ through the *line* containing $[P', Q']$ we can produce a domain $\widetilde{D_0}$ of area strictly bigger than A and perimeter of boundary of $\widetilde{D_0}$ equal to L. This is not possible. Hence $d_\kappa(O,M) = r_0$ and $\mathcal{C}_0 = \partial B_\kappa(O, r_0)$. □

Proof of Corollary 3: Let $\mathcal{C}$ be a piecewise smooth simple closed curve having m components each enclosing area $A_i > 0$. Let

$$r_i := AS_\kappa\left(\sqrt{A_i\,(4\pi - \kappa\,A_i)}/(2\pi)\right)\ (1 \leq i \leq m).$$

Let $\tilde{\mathcal{C}}$ denote the disjoint union of the circles $\tilde{\mathcal{C}}_i$ of radius r_i, $1 \leq i \leq m$. Applying Theorem 1.2 to each component of $\mathcal{C}$ we get that $perimeter(\tilde{\mathcal{C}}) \leq perimeter(\mathcal{C})$. Now, it is easy to see that a single circle with radius

$$AS_\kappa\left(\frac{\sqrt{A\,(4\pi - \kappa\,A)}}{2\pi}\right)$$

is the best. □

Remark 6.1. (1) Fix $L_0 \in (0, 2\pi]$. Put $r_0 := \arcsin(L_0/(2\pi)) \in (0, 2\pi]$ and $A_0 := 4\pi \sin^2\left(\frac{r_0}{2}\right)$. Let $\mathcal{C}$ be any piecewise smooth simple closed curve in M^2_1 having arc-length $\ell(\mathcal{C}) = L_0$. From Theorem 1.2, it follows that among all such curves, area maximizer is the circle $\mathcal{C}_{r_0}$. For, consider the family $\mathcal{F} = \{$ all piecewise smooth simple closed curves in M^2_1 enclosing area $\geq A_0\}$. If $A(\mathcal{C}) \geq A_0$, then $\mathcal{C} \in \mathcal{F}$ and $\ell(\mathcal{C}) = L_0 = \ell(\mathcal{C}_{r_0}) = \inf\{\,\ell(\tilde{\mathcal{C}})\,|\,\tilde{\mathcal{C}} \in \mathcal{F}\,\}$. By Theorem 1.2, $\mathcal{C} = \mathcal{C}_{r_0}$ and $A(\mathcal{C}) = A_0$.

(2) A shorter though less elementary approach to prove Theorem 1.1 for M^2_{-1} is to first prove Theorem 1.2 for this case and then derive the results for n-gons using Heron's formula or L'Huilier's Theorem as in [[43], Proposition 2.15].

Proof of Theorem 1.4: Let $\mathcal{C}$ be any piecewise smooth simple closed curve in M^2_κ with arc-length $\ell := \ell(\mathcal{C})$ and enclosing area $A := A(\mathcal{C}) > 0$ ($A \leq 2\pi$ if $\kappa = 1$).

Case (i) $\kappa = 1$ and $A = 2\pi$:

Let $\mathcal{J}$ and L be as in the proof of Theorem 1.2 for the corresponding case. Also recall that $L = 2\pi$. Therefore, $L^2 = 4\pi^2 = 4\pi A - A^2$ and hence $\ell^2 = [\ell(\mathcal{C})]^2 \geq L^2 = 4\pi A - A^2$ holds for all $\mathcal{C} \in \mathcal{J}$.

If for a curve $\mathcal{C}$ in $\mathcal{J}$, $[\ell(\mathcal{C})]^2 = 4\pi A - A^2 = 4\pi^2$, then

$$\ell(\mathcal{C}) = \sqrt{4\pi A - A^2} = 2\pi = L$$

and $\mathcal{C}$ is a perimeter minimizer in $\mathcal{J}$. By Theorem 1.2, $\mathcal{C}$ is a great circle i.e., a circle in M^2_1 of radius $\frac{\pi}{2} = \arcsin\left(\sqrt{A\,(4\pi - A)}/(2\pi)\right)$.

Case (ii) $\kappa \in \{-1, 0, 1\}$ and $A < 2\pi$ if $\kappa = 1$:

Let $\mathcal{G}$ and L be as in the proof of Theorem 1.2 for the corresponding case. Put $r_0 := AS_\kappa\left(\sqrt{A\,(4\pi - \kappa\, A)}/(2\pi)\right)$. By Theorem 1.2, the circle $\mathcal{C}_{r_0}$ of radius r_0 is the unique perimeter minimizer in $\mathcal{G}$. Therefore, $L^2 = (2\pi S_\kappa(r_0))^2 = 4\pi A - \kappa\, A^2$ and hence $\ell^2 = [\ell(\mathcal{C})]^2 \geq L^2 = 4\pi A - \kappa\, A^2$ holds for all $\mathcal{C}$ in $\mathcal{G}$.

If for a curve $\mathcal{C}$ in $\mathcal{G}$, $[\ell(\mathcal{C})]^2 = 4\pi A - \kappa\, A^2$, then $\ell(\mathcal{C}) = \sqrt{4\pi A - \kappa\, A^2} = L$ and $\mathcal{C}$ is a perimeter minimizer in $\mathcal{G}$. By Theorem 1.2, $\mathcal{C}$ is a circle of radius r_0 in M^2_κ. ☐

7. APPENDIX

7.1. **Appendix A.** We state some formulae about S_κ and C_κ when $\kappa \neq 0$.

$$C_\kappa(-a) = C_\kappa(a),\ S_\kappa(-a) = -S_\kappa(a). \qquad (A-1)$$

$$S_\kappa(a+b) = S_\kappa(a)\, C_\kappa(b) + C_\kappa(a)\, S_\kappa(b). \qquad (A-2)$$

$$S_\kappa(a-b) = S_\kappa(a)\, C_\kappa(b) - C_\kappa(a)\, S_\kappa(b). \qquad (A-3)$$

$$S_\kappa(2\,a) = 2\, S_\kappa(a)\, C_\kappa(a). \qquad (A-4)$$

$$C^2_\kappa(a) = 1 - \kappa\, S^2_\kappa(a). \qquad (A-5)$$

$$C_\kappa(a+b) = C_\kappa(a)\, C_\kappa(b) - \kappa\, S_\kappa(a)\, S_\kappa(b). \qquad (A-6)$$

$$C_\kappa(a-b) = C_\kappa(a)\, C_\kappa(b) + \kappa\, S_\kappa(a)\, S_\kappa(b). \qquad (A-7)$$

$$C_\kappa(2\,a) = C^2_\kappa(a) - \kappa\, S^2_\kappa(a) = 1 - 2\,\kappa\, S^2_\kappa(a) = 2\, C^2_\kappa(a) - 1. \qquad (A-8)$$

$$1 - C_\kappa(a) = 2\,\kappa\, S^2_\kappa(a/2). \qquad (A-9)$$

$$1 + C_\kappa(a) = 2\, C^2_\kappa(a/2). \qquad (A-10)$$

$$C_\kappa(a+b)+C_\kappa(a-b)=2\,C_\kappa(a)\,C_\kappa(b). \qquad (A-11)$$

$$C_\kappa(a+b)-C_\kappa(a-b)=-2\,\kappa\,S_\kappa(a)\,S_\kappa(b) \qquad (A-12)$$

$$2\,S_\kappa\left(\frac{a+b}{2}\right)C_\kappa\left(\frac{a-b}{2}\right)=S_\kappa(a)+S_\kappa(b). \qquad (A-13)$$

$$2\,C_\kappa\left(\frac{a+b}{2}\right)S_\kappa\left(\frac{a-b}{2}\right)=S_\kappa(a)-S_\kappa(b). \qquad (A-14)$$

$$2\,C_\kappa\left(\frac{a+b}{2}\right)C_\kappa\left(\frac{a-b}{2}\right)=C_\kappa(a)+C_\kappa(b). \qquad (A-15)$$

$$-2\,\kappa\,S_\kappa\left(\frac{a+b}{2}\right)S_\kappa\left(\frac{a-b}{2}\right)=C_\kappa(a)-C_\kappa(b). \qquad (A-16)$$

7.2. **Appendix B - Trigonometric formulae for a triangle in M_κ^2.** Let $[P,Q,R]$ be a triangle in M_κ^2 having angles α,β,γ at its vertices and let a,b,c be sides opposite to angles α,β,γ, respectively. Put $s=\dfrac{a+b+c}{2}$. Then we have following formulae:

$$\sin\frac{\gamma}{2}=\sqrt{\frac{S_\kappa(s-a)S_\kappa(s-b)}{S_\kappa(a)S_\kappa(b)}}. \qquad (B-1)$$

$$\cos\frac{\gamma}{2}=\sqrt{\frac{S_\kappa(s)\,S_\kappa(s-c)}{S_\kappa(a)\,S_\kappa(b)}}. \qquad (B-2)$$

The sine rule:

$$\frac{\sin\alpha}{S_\kappa(a)}=\frac{\sin\beta}{S_\kappa(b)}=\frac{\sin\gamma}{S_\kappa(c)}=\frac{2\sqrt{S_\kappa(s)\,S_\kappa(s-a)S_\kappa(s-b)S_\kappa(s-c)}}{S_\kappa(a)\,S_\kappa(b)\,S_\kappa(c)}. \qquad (B-3)$$

The following holds when $\kappa\neq 0$:

$$\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\frac{\gamma}{2}\;\frac{C_\kappa\left(\frac{a-b}{2}\right)}{C_\kappa\left(\frac{c}{2}\right)}. \qquad (B-4)$$

$$\sin\left(\frac{\alpha-\beta}{2}\right)=\cos\frac{\gamma}{2}\;\frac{S_\kappa\left(\frac{a-b}{2}\right)}{S_\kappa\left(\frac{c}{2}\right)}. \qquad (B-5)$$

$$\cos\left(\frac{\alpha+\beta}{2}\right)=\sin\frac{\gamma}{2}\;\frac{C_\kappa\left(\frac{a+b}{2}\right)}{C_\kappa\left(\frac{c}{2}\right)}. \qquad (B-6)$$

$$\cos\left(\frac{\alpha-\beta}{2}\right)=\sin\frac{\gamma}{2}\;\frac{S_\kappa\left(\dfrac{a+b}{2}\right)}{S_\kappa\left(\dfrac{c}{2}\right)}. \qquad (B-7)$$

Proof of (B-1): We give the proof of (B-1) for triangles in M^2_κ ($\kappa\neq 0$). The proof for triangles in M^2_0 is similar and simpler.

By the Law of Cosine for triangles in M^2_κ ($\kappa\neq 0$) we have

$$\cos(\gamma)=\frac{C_\kappa(c)-C_\kappa(a)\,C_\kappa(b)}{\kappa\,S_\kappa(a)\,S_\kappa(b)}. \qquad (B-8)$$

By (A-9),

$$\begin{aligned}
2\sin^2\frac{\gamma}{2} &= 1-\frac{C_\kappa(c)-C_\kappa(a)\,C_\kappa(b)}{\kappa\,S_\kappa(a)\,S_\kappa(b)}=\frac{\kappa\,S_\kappa(a)\,S_\kappa(b)+C_\kappa(a)\,C_\kappa(b)-C_\kappa(c)}{\kappa\,S_\kappa(a)\,S_\kappa(b)}\\
&=\frac{C_\kappa(a-b)-C_\kappa(c)}{\kappa\,S_\kappa(a)\,S_\kappa(b)}\ \text{[by (A-7)]}\\
&=\frac{2\kappa\,S_\kappa\left(\dfrac{c+a-b}{2}\right)S_\kappa\left(\dfrac{c-a+b}{2}\right)}{\kappa\,S_\kappa(a)\,S_\kappa(b)}\ \text{[by (A-16) and (A-1)]}\\
&=\frac{2S_\kappa(s-b)S_\kappa(s-a)}{S_\kappa(a)\,S_\kappa(b)}.
\end{aligned}$$

Now (B-1) follows easily.

Proof of (B-2): We give the proof of (B-2) for triangles in M^2_κ ($\kappa\neq 0$). The proof for triangles in M^2_0 is similar and simpler.

By (B-8) and (A-10) we have:

$$\begin{aligned}
2\cos^2\frac{\gamma}{2} &= 1+\frac{C_\kappa(c)-C_\kappa(a)\,C_\kappa(b)}{\kappa\,S_\kappa(a)\,S_\kappa(b)}\\
&=\frac{-C_\kappa(a)\,C_\kappa(b)+\kappa\,S_\kappa(a)\,S_\kappa(b)+C_\kappa(c)}{\kappa\,S_\kappa(a)\,S_\kappa(b}\\
&=\frac{-C_\kappa(a+b)+C_\kappa(c)}{\kappa\,S_\kappa(a)\,S_\kappa(b)}\ \text{[by (A-6)]}\\
&=\frac{2\kappa\,S_\kappa\left(\dfrac{a+b+c}{2}\right)S_\kappa\left(\dfrac{a+b-c}{2}\right)}{\kappa\,S_\kappa(a)\,S_\kappa(b)}\ \text{[by (A-16) and (A-1)]}\\
&=\frac{2S_\kappa(s)\,S_\kappa(s-c)}{S_\kappa(a)\,S_\kappa(b)}.
\end{aligned}$$

Now (B-2) follows easily.

Proof of (B-3): By (A-4), (B-1) and (B-2) we get,

$$\frac{\sin\gamma}{S_\kappa(c)} = \frac{2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}}{S_\kappa(c)} = \frac{2\sqrt{S_\kappa(s)\,S_\kappa(s-a)S_\kappa(s-b)S_\kappa(s-c)}}{S_\kappa(a)\,S_\kappa(b)\,S_\kappa(c)}.$$

Hence,

$$\frac{\sin\alpha}{S_\kappa(a)} = \frac{\sin\beta}{S_\kappa(b)} = \frac{\sin\gamma}{S_\kappa(c)} = \frac{2\sqrt{S_\kappa(s)\,S_\kappa(s-a)S_\kappa(s-b)S_\kappa(s-c)}}{S_\kappa(a)\,S_\kappa(b)\,S_\kappa(c)}.$$

Proof of (B-4): By (A-2), (B-1) and (B-2) we get,

$$\begin{aligned}
\sin\left(\frac{\alpha+\beta}{2}\right) &= \sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2} \\
&= \sqrt{\frac{S_\kappa(s-b)S_\kappa(s-c)}{S_\kappa(b)\,S_\kappa(c)}}\sqrt{\frac{S_\kappa(s)\,S_\kappa(s-b)}{S_\kappa(a)\,S_\kappa(c)}} \\
&\quad + \sqrt{\frac{S_\kappa(s)\,S_\kappa(s-a)}{S_\kappa(b)\,S_\kappa(c)}}\sqrt{\frac{S_\kappa(s-a)S_\kappa(s-c)}{S_\kappa(a)\,S_\kappa(c)}} \\
&= \sqrt{\frac{S_\kappa(s)\,S_\kappa(s-c)}{S_\kappa(a)\,S_\kappa(b)}}\left(\frac{S_\kappa(s-b)+S_\kappa(s-a)}{S_\kappa(c)}\right) \\
&= \cos\left(\frac{\gamma}{2}\right)\left(\frac{S_\kappa(s-b)+S_\kappa(s-a)}{S_\kappa(c)}\right) \text{ [by (B-2)]} \\
&= \cos\left(\frac{\gamma}{2}\right)\frac{2S_\kappa\left(\frac{2s-a-b}{2}\right)C_\kappa\left(\frac{a-b}{2}\right)}{S_\kappa(c)} \\
&\qquad \text{[by (A-13), (A-1)]} \\
&= \cos\left(\frac{\gamma}{2}\right)\frac{2S_\kappa\left(\frac{c}{2}\right)C_\kappa\left(\frac{a-b}{2}\right)}{2\,S_\kappa\left(\frac{c}{2}\right)C_\kappa\left(\frac{c}{2}\right)} \text{ [by (A-4)]} \\
&= \cos\left(\frac{\gamma}{2}\right)\frac{C_\kappa\left(\frac{a-b}{2}\right)}{C_\kappa\left(\frac{c}{2}\right)}.
\end{aligned}$$

Proof of (B-5): Similarly, by (A-3), (B-1) and (B-2) we get,

$$\begin{aligned}
\sin\left(\frac{\alpha-\beta}{2}\right) &= sin\frac{\alpha}{2}\cos\frac{\beta}{2}-\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\\
&= \cos\left(\frac{\gamma}{2}\right)\left(\frac{S_\kappa(s-b)-S_\kappa(s-a)}{S_\kappa(c)}\right)\\
&= \cos\left(\frac{\gamma}{2}\right)\frac{2C_\kappa\left(\frac{c}{2}\right)S_\kappa\left(\frac{a-b}{2}\right)}{2S_\kappa\left(\frac{c}{2}\right)C_\kappa\left(\frac{c}{2}\right)} \text{ [by (A-4) and (A-14)]}\\
&= \cos\left(\frac{\gamma}{2}\right)\frac{S_\kappa\left(\frac{a-b}{2}\right)}{S_\kappa\left(\frac{c}{2}\right)}.
\end{aligned}$$

Proof of (B-6): By (A-6), (B-1) and (B-2) we get,

$$\begin{aligned}
\cos\left(\frac{\alpha+\beta}{2}\right) &= \cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\\
&= \sqrt{\frac{S_\kappa(s)\,S_\kappa(s-a)}{S_\kappa(b)\,S_\kappa(c)}}\sqrt{\frac{S_\kappa(s)\,S_\kappa(s-b)}{S_\kappa(a)\,S_\kappa(c)}}\\
&\quad -\sqrt{\frac{S_\kappa(s-b)S_\kappa(s-c)}{S_\kappa(b)\,S_\kappa(c)}}\sqrt{\frac{S_\kappa(s-a)S_\kappa(s-c)}{S_\kappa(a)\,S_\kappa(c)}}\\
&= \sqrt{\frac{S_\kappa(s-a)S_\kappa(s-b)}{S_\kappa(a)\,S_\kappa(b)}}\left(\frac{S_\kappa(s)\,-S_\kappa(s-c)}{S_\kappa(c)}\right)\\
&= \sin\frac{\gamma}{2}\left(\frac{S_\kappa(s)\,-S_\kappa(s-c)}{S_\kappa(c)}\right) \text{ [by (B-1)]}\\
&= \sin\frac{\gamma}{2}\frac{2C_\kappa\left(\frac{2s-c}{2}\right)S_\kappa\left(\frac{c}{2}\right)}{2S_\kappa\left(\frac{c}{2}\right)C_\kappa\left(\frac{c}{2}\right)} \text{ [by (A-4) and (A-14)]}\\
&= \sin\frac{\gamma}{2}\frac{C_\kappa\left(\frac{a+b}{2}\right)}{C_\kappa\left(\frac{c}{2}\right)}.
\end{aligned}$$

Proof of (B-7): Similarly, by (A-17), (B-1), (B-2), (A-4), (A-13) and (A-1) we get

$$\begin{aligned}\cos\left(\frac{\alpha-\beta}{2}\right) &= \cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2} = \sin\frac{\gamma}{2}\left(\frac{S_\kappa(s)+S_\kappa(s-c)}{S_\kappa(c)}\right)\\ &= \sin\frac{\gamma}{2}\,\frac{2S_\kappa\left(\frac{2s-c}{2}\right)C_\kappa\left(\frac{c}{2}\right)}{2S_\kappa\left(\frac{c}{2}\right)C_\kappa\left(\frac{c}{2}\right)} = \sin\frac{\gamma}{2}\,\frac{S_\kappa\left(\frac{a+b}{2}\right)}{S_\kappa\left(\frac{c}{2}\right)}.\ \square\end{aligned}$$

7.3. **Appendix C - Proof of Proposition 3.8.**

(i) $\Longrightarrow$ (ii) Let f be an isometry of M_κ^2 such that $f(P)=P'$, $f(Q)=Q'$ and $f(R)=R'$. Then, $a := d_\kappa(P,Q) = d_\kappa(f(P),f(Q)) = d_\kappa(P',Q') =: a'$. Similarly, $b=b'$ and $c=c'$.

(ii) $\Longrightarrow$ (i) This follows immediately from Proposition 2.4.

(ii) $\Longrightarrow$ (iii) By the Law of Cosine it follows that

$\underline{\kappa=0}$:

$$\cos\alpha = \frac{b^2+c^2-a^2}{2\,b\,c} = \frac{(b')^2+(c')^2-(a')^2}{2\,b'\,c'} = \cos\alpha'.$$

$\underline{\kappa\neq 0}$:

$$\cos\alpha = \frac{C_\kappa(a)-C_\kappa(b)\,C_\kappa(c)}{\kappa\,S_\kappa(b)\,S_\kappa(c)} = \frac{C_\kappa(a')-C_\kappa(b')\,C_\kappa(c')}{\kappa\,S_\kappa(b')\,S_\kappa(c')} = \cos\alpha'.$$

As $\alpha,\alpha'\in(0,\pi)$ we get $\alpha=\alpha'$.

(iii) $\Longrightarrow$ (ii) By the Law of Cosine we have

$\underline{\kappa=0}$:

$$a^2 = b^2+c^2-2\,b\,c\,\cos\alpha = (b')^2+(c')^2-2\,b'\,c'\,\cos\alpha' = (a')^2.$$

Since $a'>0$ we get $a=a'$.

$\underline{\kappa\neq 0}$:

$$\begin{aligned}C_\kappa(a) &= C_\kappa(b)\,C_\kappa(c)+\kappa\,S_\kappa(b)\,S_\kappa(c)\,\cos\alpha\\ &= C_\kappa(b')\,C_\kappa(c')+\kappa\,S_\kappa(b')\,S_\kappa(c')\,\cos\alpha'\\ &= C_\kappa(a')\end{aligned}$$

Let

$$I'_\kappa := \begin{cases} (0,\pi) & \text{if } \kappa = 1, \\ (0,\infty) & \text{if } \kappa = -1. \end{cases}$$

Since $a, a' \in I'_\kappa$ we get $a = a'$.
Similarly we can prove that $b = b'$ and $c = c'$.

(iii) $\Longrightarrow$ (iv) As (iii) $\Longrightarrow$ (ii) we have $a = a'$, $b = b'$, $c = c'$ and $\alpha = \alpha'$. By the Law of Cosine it follows that $\beta = \beta'$ and $\gamma = \gamma'$.

(iv) $\Longrightarrow$ (iii) By the Law of Sine applied to the triangles T and T' we have,

$$\frac{S_\kappa(a)}{\sin\alpha'} = \frac{S_\kappa(b')}{\sin\beta} = \frac{S_\kappa(c')}{\sin\gamma}.$$

and

$$\frac{S_\kappa(a)}{\sin\alpha} = \frac{S_\kappa(b)}{\sin\beta} = \frac{S_\kappa(c)}{\sin\gamma}.$$

Hence,

$$\frac{S_\kappa(b)}{S_\kappa(c)} = \frac{\sin\beta}{\sin\gamma} = \frac{S_\kappa(b')}{S_\kappa(c')}.$$

Therefore

$$S_\kappa(b)\, S_\kappa(c') = S_\kappa(b')\, S_\kappa(c). \qquad (C-1)$$

By the Law of Cosine we have,
<u>$\kappa = 0$</u>:

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2\,a\,b} = \frac{2\,a^2 - 2\,a\,c\,\cos\beta}{2\,a\,b} = \frac{a - c\,\cos\beta}{b}.$$

Similarly, $\cos\gamma' = \dfrac{a' - c'\,\cos\beta'}{b'}$. From the hypothesis it follows that $\dfrac{a - c\,\cos\beta}{b} = \dfrac{a - c'\,\cos\beta}{b'}$. Therefore, $a\,(b' - b) = \cos\beta\,(b'\,c - b\,c')$. Thus from (C-1) it follows that $b = b'$. Now, $a = a'$, $b = b'$ and $\gamma = \gamma'$ is another form of (iii).

$\underline{\kappa \neq 0}$:

$$\begin{aligned}
\cos\gamma &= \frac{C_\kappa(c) - C_\kappa(a)\, C_\kappa(b)}{\kappa\, S_\kappa(a)\, S_\kappa(b)} \\
&= \frac{C_\kappa(c) - C_\kappa(a)\ [C_\kappa(a)\, C_\kappa(c) + \kappa\, S_\kappa(a)\, S_\kappa(c)\ \cos\beta]}{\kappa\, S_\kappa(a)\, S_\kappa(b)} \\
&= \frac{\kappa\, C_\kappa(c)\, S_\kappa^2(a) + \kappa\, S_\kappa(a)\, C_\kappa(a)\, S_\kappa(c)\ \cos\beta}{\kappa\, S_\kappa(a)\, S_\kappa(b)} \\
&\quad [\text{by (A-5)}].
\end{aligned}$$

Since $a \in I'_\kappa$, $S_\kappa(a) \neq 0$. Therefore we get

$$\cos\gamma = \frac{C_\kappa(c)\, S_\kappa(a) + C_\kappa(a)\, S_\kappa(c)\ \cos\beta}{S_\kappa(b)}.$$

Similar calculations on Triangle T' yields

$$\begin{aligned}
\cos\gamma' &= \frac{C_\kappa(c')\, S_\kappa(a') + C_\kappa(a')\, S_\kappa(c')\ \cos\beta'}{S_\kappa(b')} \\
&= \frac{C_\kappa(c')\, S_\kappa(a) + C_\kappa(a)\, S_\kappa(c')\ \cos\beta}{S_\kappa(b')}.
\end{aligned}$$

Since $\gamma = \gamma'$ we get $S_\kappa(b')\,[C_\kappa(c)\, S_\kappa(a) + C_\kappa(a)\, S_\kappa(c)\ \cos\beta]$ $= S_\kappa(b)\,[C_\kappa(c')\, S_\kappa(a) + C_\kappa(a)\, S_\kappa(c')\ \cos\beta]$. That is,

$$\begin{aligned}
& S_\kappa(a)\ \left[S_\kappa(b)\, C_\kappa(c') - S_\kappa(b')\, C_\kappa(c)\right] \\
= \ & -C_\kappa(a)\ \cos\beta\ \left[S_\kappa(b)\, S_\kappa(c') - S_\kappa(b')\, S_\kappa(c)\right]. \quad (C-2)
\end{aligned}$$

As $S_\kappa(a) \neq 0$, from (C-1) and (C-2) it follows that $S_\kappa(b)\, C_\kappa(c') - S_\kappa(b')\, C_\kappa(c) = 0$. That is,

$$\frac{S_\kappa(b)}{S_\kappa(b')} = \frac{C_\kappa(c)}{C_\kappa(c')}. \qquad (C-3)$$

From (C-1) and (C-3) we get $T_\kappa(c) = T_\kappa(c')$. As $c, c' \in I'_\kappa$ we get $c = c'$. Now, $a = a'$, $c = c'$ and $\beta = \beta'$ is another form of (iii).

(ii) $\Longrightarrow$ (v) In the proof of (ii) $\Longrightarrow$ (iii), we showed that $\alpha = \alpha'$. Similarly it can be proved that $\beta = \beta'$ and $\gamma = \gamma'$.

(v) $\Longrightarrow$ (iv) when $\boldsymbol{\kappa \neq 0}$: Let A, A' denote areas of triangles T, T' respectively. By Theorem 3.2,
$A = \kappa\ (\pi - \alpha + \beta + \gamma) = \kappa\ (\pi - \alpha' + \beta' + \gamma') = A'$. Therefore, by Proposition 3.6 it follows that

$$\frac{CT_\kappa\left(\frac{a}{2}\right)\, CT_\kappa\left(\frac{b}{2}\right) + \kappa\,\cos\gamma}{\sin\gamma} = \frac{CT_\kappa\left(\frac{a'}{2}\right)\, CT_\kappa\left(\frac{b'}{2}\right) + \kappa\,\cos\gamma'}{\sin\gamma'}.$$

Since $\gamma = \gamma'$ we get

$$CT_\kappa\left(\frac{a}{2}\right)\, CT_\kappa\left(\frac{b}{2}\right) = CT_\kappa\left(\frac{a'}{2}\right)\, CT_\kappa\left(\frac{b'}{2}\right). \qquad (C-4)$$

Similarly we have

$$CT_\kappa\left(\frac{a}{2}\right)\, CT_\kappa\left(\frac{c}{2}\right) = CT_\kappa\left(\frac{a'}{2}\right)\, CT_\kappa\left(\frac{c'}{2}\right), \qquad (C-5)$$

and

$$CT_\kappa\left(\frac{b}{2}\right)\, CT_\kappa\left(\frac{c}{2}\right) = CT_\kappa\left(\frac{b'}{2}\right)\, CT_\kappa\left(\frac{c'}{2}\right). \qquad (C-6)$$

Multiplying (C-4) and (C-5) we get

$$CT_\kappa^2\left(\frac{a}{2}\right)\, CT_\kappa\left(\frac{b}{2}\right)\, CT_\kappa\left(\frac{c}{2}\right) = CT_\kappa^2\left(\frac{a'}{2}\right)\, CT_\kappa\left(\frac{b'}{2}\right)\, CT_\kappa\left(\frac{c'}{2}\right).$$

As $CT_\kappa(x) \neq 0$ for $x \in I'_\kappa$, by (C-6) we get $CT_\kappa^2\left(\frac{a}{2}\right) = CT_\kappa^2\left(\frac{a'}{2}\right)$. That is,

$$T_\kappa^2\left(\frac{a}{2}\right) = T_\kappa^2\left(\frac{a'}{2}\right). \qquad (C-7)$$

As $a, a' \in I'_\kappa$, $\frac{a}{2}, \frac{a'}{2} \in I''_\kappa$ where $I''_\kappa = \begin{cases} (0, \frac{\pi}{2}) & \text{if } \kappa = 1 \\ (0, \infty) & \text{if } \kappa = -1. \end{cases}$ Therefore, $T_\kappa(a),\ T_\kappa(a') > 0$ and, hence from (C-7) we get $T_\kappa(a) = T_\kappa(a')$. Finally, since T_κ is an injective function on I''_κ we get $a = a'$.

Similarly it can be shown that $b = b'$ and $c = c'$. Thus, in fact we have (v) $\Longrightarrow$ (ii).

This completes the proof of Proposition 3.8. □

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ON CERTAIN IDENTITIES FOR HYPERGEOMETRIC SERIES

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Abstract. We study an elementary hypergeometric ${}_3F_2\left(\frac{1}{4}\right)$ series studied by Ramanujan in his second notebook. We derive a closed-form for this series in terms of the Barnes G-function and $\log(\pi)$. In the process, using the integrals $\int_0^{\frac{\pi}{4}} \ln^2(\sin q)\, dq$ and $\int_0^{\frac{\pi}{4}} \ln(\sin q)\, dq$ we give an elementary proof of Finch's hypergeometric ${}_4F_3\left(\frac{1}{2}\right)$ series due to Sofo and Nimbran (Integral Transforms Spec Funct. **31** (2020) 966-981), where they have proved using the (BBP-type) identity. Further, using the recent result of Campbell we solve an open problem considered by Steven Finch, as far back as 2007, concerning the computing a closed-form for the hypergeometric ${}_3F_2$ function. Finally, we offer explicitly an intergral representation for the hypergeometric ${}_3F_2(-1)$ series, alternating version of the Ramanujan's hypergeometric ${}_3F_2(1)$ series, also considered as an open problem by Steven Finch in 2007.

1. Introduction

On page 106 of his second notebook [9, Chap. IX, p. 106, entry 16. example iii], Ramanujan studied the hypergeometric ${}_3F_2\left(\frac{1}{4}\right)$ function

$$ {}_3F_2\left(\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{array} \middle| \frac{1}{4}\right) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n} 2^{-4n}}{(2n+1)^2} = \frac{3\sqrt{3}}{2} \sum_{n=0}^{\infty} \frac{1}{(3n+1)^2} - \frac{\pi^2}{3\sqrt{3}}. \quad (1.1) $$

Among other things Ramanujan studied the identities

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$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}(2n+1)^2} = \frac{\pi}{2}\log(2)$$

and

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{3n}(2n+1)^2} = \frac{\pi}{4\sqrt{2}}\log(2) + \frac{G}{\sqrt{2}}. \tag{1.2}$$

The series in (1.2) has been studied recently by the author of this paper in his article [20] who derive a new evaluation for it.

In this paper, we give an elementary proof of the identity (1.1) mentioned above. We express the series in closed form using the logarithm of the transcendental number, π, and the the double gamma function (or Barnes G-function), $\Gamma_2(z) = 1/G(z)$, the later being defined by [4],[5],[6],[7] (see also [8, pp. 94-96])

$$(\Gamma_2(z+1))^{-1} = (2\pi)^{z/2} e^{\left(-\frac{z}{2} - \frac{(\gamma+1)z^2}{2}\right)} \prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^k e^{\left(-z+\frac{z^2}{2k}\right)}\right],$$

where $\gamma = 0.57721...$ is the Euler-Mascheroni constant.

Berndt [10, p. 264] proved (1.1) using the method of substitution and Zucker [23, Equation (2.13)] expressed the series (1.1) in closed form using the identity

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^{2k}(2k+1)^i} = \frac{(-1)^{i-1}}{(i-1)!}\int_0^{\frac{\pi}{3}} \log^{i-1}\left(2\sin\frac{\phi}{2}\right) d\phi,$$

in terms of Dirichlet L-series, $L_d(s)$, defined by

$$L_d(s) = \sum_{k=1}^{\infty} \chi_d(k) k^{-s},$$

where $\chi_d(k)$ is a character modulo d.

In addition, we will offer an alternative rigorous solution for the hypergeometric ${}_4F_3\left(\frac{1}{2}\right)$ series, posed as an open problem by Finch in [1, p. 5], reads explicitly as:

$$ {}_4F_3\left(\begin{array}{c} \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ \frac{3}{2},\frac{3}{2},\frac{3}{2} \end{array} \middle| \frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{3n}\,(2n+1)^3}. \tag{1.3} $$

The symbolic evaluation for this difficult hypergeometric series in (1.3) was given implicitly in [16, p. 979] by Sofo and Nimbran using the (BBP-type) identity. An alternate and explicit evaluation for the series in (1.3) was offered in [17, pp. 129-130] by Campbell, Levrie and Nimbran.

Further in this paper, we give closed-form expression for the following hypergeometric ${}_3F_2\left(-\frac{1}{2}\right)$ series, also left as an open problem by Finch in [1, p. 5]:

$$ {}_3F_2\left(\begin{array}{c} \frac{1}{2},1,1 \\ \frac{3}{2},\frac{3}{2} \end{array} \middle| -\frac{1}{2}\right) := \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{\binom{2n}{n}(2n+1)^2}. \tag{1.4} $$

Finally, we offer an intergral representation for the following alternating version of the Ramanujan's hypergeometric ${}_3F_2\,(1)$ series, also left as an open problem by Finch in [1, p. 5]:

$$ {}_3F_2\left(\begin{array}{c} \frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 1,\frac{3}{2} \end{array} \middle| -1\right) := \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^2}{2^{4n}\,(2n+1)}. \tag{1.5} $$

The closed-form expression for the nonalternating version of the hypergeometric series in (1.5), ${}_3F_2\left(\begin{array}{c} \frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 1,\frac{3}{2} \end{array} \middle| 1\right) = \sum\limits_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)} = \frac{4G}{\pi}$, had been recorded by Ramanujan on page 123 in his second notebook [9, Chap. X, p. 123, entry 29. corollary 1]. The proofs for this nonalternating version can be traced through the articles [11, Equation (29.3), p. 40],[12, Equation (2.4)], [14, Thm. 13, p. 17] and [21]. Hypergeometric series, similar to this nonalternating version, but a more advanced one where the difference is that in denominator contains $(2n+1)^2$ instead of $(2n+1)$, ${}_4F_3\left(\begin{array}{c} \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 1,\frac{3}{2},\frac{3}{2} \end{array} \middle| 1\right) = \sum\limits_{n=0}^{\infty} \frac{\binom{2n}{n}^2}{2^{4n}(2n+1)^2}$, was considered in *Mathematics Student* article [22] in 2015 by Nimbran and closed-form expression in terms of $\mathcal{G}$ for Nimbran's ${}_4F_3\left(1\right)$ series was given in [18, Equation (13)], [19, Equation (32)].

While the closed-form expression for (1.5) seems to be out of reach by present methods, we provide an intergral representation for (1.5) below.

We recall that the (generalized) hypergeometric function may be defined to be the complex analytic function so that [3], for $p, q \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $a_1, \ldots, a_p, b_1, \ldots, b_q, x \in \mathbb{C}$,

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \middle| \, x \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{x^n}{n!} $$

where $(a)_k = a \cdot (a+1) \cdots (a+k-1)$ for $k \in \mathbb{N}$ denotes the Pochhammer symbol.

Throughout this paper, we use the same notations and definitions as [15], letting $\binom{2n}{n}$ denotes the central binomial coefficient, which is defined for $n \geq 1$ by $\binom{2n}{n} = (2n)!/\left(n!\right)^2$, $\zeta\left(s\right)$ denotes the Riemann zeta function, which is defined by $\zeta\left(s\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1$, G denotes the Catalan's constant, which is defined by $G = \frac{1}{2} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, 1, 1 \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \middle| \, 1 \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2}$ and $\mathcal{G}$ denotes the Catalan-like constant, which is defined by $\mathcal{G} = \Im\left(\mathrm{Li}_3\left(\frac{i+1}{2}\right)\right)$.

In the following, $\csc^{-1}\left(x\right)$ denotes the inverse cosecant, which is defined for $x \in (-\infty, -1] \cup [1, \infty)$ by $\csc^{-1}\left(x\right) = -\iota \ln\left(\sqrt{1 - \frac{1}{x^2}} + \frac{\iota}{x}\right)$, $\sinh^{-1}\left(x\right)$ denotes the inverse hyperbolic sine, which is defined for $x \in (-\infty, \infty)$ by $\sinh^{-1}\left(x\right) = \ln\left(x + \sqrt{1 + x^2}\right)$, $\sin^{-1}\left(x\right)$ denotes the inverse sine, which is defined for $x \in [-1, 1]$ by $\sin^{-1}\left(x\right) = -\iota \ln\left(\iota x + \sqrt{1 - x^2}\right)$, with $\sin^{-1}\left(\iota x\right) = \iota \sinh^{-1}\left(x\right)$ and $\mathrm{Li}_n(x)$ denotes the polylogarithm function, which is defined for $|x| \leq 1$ by $\mathrm{Li}_{\mathrm{n}}\left(x\right) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}, \quad n \in \mathbb{N}, \quad n \geq 2$.

To evaluate (1.1), (1.3) and (1.5), we shall establish some lemmas.

Lemma 1.1. *Let* $\alpha > -\frac{1}{2}$. *The following equality holds:*

$$\int_0^1 x^{2\alpha} \ln(x)\, dx = -\frac{1}{(2\alpha+1)^2}.$$

Proof. The proof is given in the paper [13]. □

Lemma 1.2. *The following integral formula holds true:*

$$\int_0^{\frac{\pi}{6}} \ln(\sin t)\, dt = -\frac{\pi}{6}\ln\pi - \frac{\pi}{3}\ln 2 + \pi \ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right).$$

Proof. In the article [2, Equation (4.5)], Choi, Cho and Srivastava proved that

$$\int_0^{\frac{\pi}{3}} \ln\left(2\sin\frac{v}{2}\right) dv = -\frac{\pi}{3}\ln(2\pi) + 2\pi\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right) \tag{1.6}$$

We make the substitution $v = 2t$ on the left side of (1.6), so that $\frac{dv}{dt} = 2$

Then,
$$\begin{aligned}\int_0^{\frac{\pi}{3}} \ln\left(2\sin\frac{v}{2}\right) dv &= 2\int_0^{\frac{\pi}{6}} \ln(2\sin t)\, dt \\ &= 2\left(\int_0^{\frac{\pi}{6}} \ln(2)\, dt + \int_0^{\frac{\pi}{6}} \ln(\sin t)\, dt\right) \\ &= \frac{\pi}{3}\ln 2 + 2\int_0^{\frac{\pi}{6}} \ln(\sin t)\, dt \\ &= -\frac{\pi}{3}\ln(2\pi) + 2\pi\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right)\end{aligned}$$

giving us that the following equality holds:

$$\begin{aligned}\int_0^{\frac{\pi}{6}} \ln(\sin t)\, dt &= -\frac{\pi}{6}\ln(2\pi) - \frac{\pi}{6}\ln 2 + \pi\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right) \\ &= -\frac{\pi}{6}\ln\pi - \frac{\pi}{3}\ln 2 + \pi\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right).\end{aligned}$$

□

Lemma 1.3. *The following integral formula holds true:*

$$\int_0^{\frac{\pi}{4}} \ln(\sin q)\, dq = -\frac{\pi}{4}\ln 2 - \frac{G}{2}.$$

Proof. The proof is detailed in the paper [15, Proof of Lemma 1.2].

□

Lemma 1.4. *The following integral formula holds true:*

$$\int_0^{\frac{\pi}{4}} \ln^2(\sin q)\, dq = \frac{9\pi}{32}\ln^2 2 + \frac{G\ln 2}{2} + \frac{23\pi^3}{384} - \mathcal{G}.$$

Proof. The proof is detailed in [15, Proof of Lemma 1.1].

□

Lemma 1.5. *The following identity holds:*

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n} x^{2n}}{2n+1} = \frac{1}{2x}\sin^{-1}(2x).$$

Proof. The proof is detailed in [21, Proof of Theorem 1.1(Step One of the First Proof)].

□

Theorem 1.6. *The evaluation*

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{4n}(2n+1)^2} = \frac{\pi}{3}\ln(\pi) + \frac{\pi}{3}\ln(2) - 2\pi\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right),$$

holds true.

Proof.
$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{4n}(2n+1)^2} &= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{4n}}\left(-\int_0^1 x^{2n}\ln(x)\, dx\right) \\ &= -\int_0^1 \ln(x)\left[\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{x^2}{16}\right)^n\right] dx\end{aligned}$$

Invoking the well-known generating function of the central binomial coefficient, $\sum_{n=0}^{\infty}\binom{2n}{n}k^n = \frac{1}{\sqrt{1-4k}}, \quad |k| < \frac{1}{4}$. we have,

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{3n}(2n+1)^4} = -2\int_0^1 \frac{\ln(x)}{\sqrt{4-x^2}}dx$$

We make the substitution: $x = 2\sin t$, so that $\frac{dx}{dt} = 2\cos t$

$$\text{Then, } -2\int_0^1 \frac{\ln(x)}{\sqrt{4-x^2}}dx = -2\int_0^{\frac{\pi}{6}} \ln(2\sin t)\,dt$$

$$= \frac{\pi}{3}\ln(\pi) + \frac{\pi}{3}\ln(2) - 2\pi\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right),$$

and the theorem is proved.

□

Theorem 1.7. *Finch's* ${}_4F_3\left(\frac{1}{2}\right)$ *series in* (1.3) *admits the evaluation*

$$\frac{23\sqrt{2}\pi^3}{768} + \frac{3\sqrt{2}\pi}{64}\ln^2(2) - \frac{\sqrt{2}}{2}\mathcal{G}.$$

Proof. Using the well-known definite integral formula $\int_0^1 x^{2n}\ln^2(x)\,dx = \frac{2}{(2n+1)^3}$, we begin with the following equalities,

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{3n}(2n+1)^3} = \sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{3n}}\left(\frac{1}{2}\int_0^1 x^{2n}\ln^2(x)\,dx\right)$$

$$= \frac{1}{2}\int_0^1 \ln^2(x)\left[\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{x^2}{8}\right)^n\right]dx$$

$$= \frac{\sqrt{2}}{2}\int_0^1 \frac{\ln^2(x)}{\sqrt{2-x^2}}dx$$

We make the substitution: $x = \sqrt{2}\sin q$, so that $\frac{dx}{dq} = \sqrt{2}\cos q$

It follows that,

$$\frac{\sqrt{2}}{2}\int_0^1 \frac{\ln^2(x)}{\sqrt{2-x^2}}dx = \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}}\ln^2\left(\sqrt{2}\sin q\right)dq$$

$$= \frac{\sqrt{2}}{2}\left[\int_0^{\frac{\pi}{4}}\ln^2\sqrt{2}dt + \int_0^{\frac{\pi}{4}}\ln^2(\sin q)\,dq\right] +$$

$$\sqrt{2}\ln\sqrt{2}\left[\int_0^{\frac{\pi}{4}}\ln(\sin q)\,dq\right]$$

$$= \frac{\sqrt{2}}{2}\left[\frac{\pi}{16}\ln^2(2) + \frac{9\pi}{32}\ln^2 2 + \frac{23\pi^3}{384} - \mathcal{G} - \frac{\pi}{4}\ln^2 2\right]$$

$$= \frac{23\sqrt{2}\pi^3}{768} + \frac{3\sqrt{2}\pi}{64}\ln^2(2) - \frac{\sqrt{2}}{2}\mathcal{G},$$

and the theorem is proved.

Here in the proofs of Theorem 1.6 and Theorem 1.7, Bernstein's theorem [25, Thm. 9.30, p. 243] justifies interchanging the order of integration and summation because of the positivity of the coefficients.

□

Theorem 1.8. *Finch's series in* (1.4) *evaluates to*

$$\sqrt{2}\,\mathrm{Li}_2\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)-\sqrt{2}\,\mathrm{Li}_2\left(\frac{-1+\sqrt{3}}{\sqrt{2}}\right)+$$
$$\frac{3\sqrt{2}}{2}\zeta(2)-\frac{\sqrt{2}}{2}\ln\left(\sqrt{3}-\sqrt{2}\right)\ln\left(2-\sqrt{3}\right).$$

Proof. In the recent article [24, p. 34], Campbell had listed the following generating function involving reciprocal of central binomial coefficient:

$$\sum_{n=0}^{\infty}\frac{x^n}{\binom{2n}{n}(2n+1)^2}=\frac{2\iota\,\mathrm{Li}_2\left(-\sqrt{1-\frac{x}{4}}-\frac{\iota\sqrt{x}}{2}\right)}{\sqrt{x}}-\frac{2\iota\,\mathrm{Li}_2\left(\sqrt{1-\frac{x}{4}}+\frac{\iota\sqrt{x}}{2}\right)}{\sqrt{x}}+$$
$$\frac{\iota\pi^2}{2\sqrt{x}}+\frac{2\ln\left(\frac{-\sqrt{1-\frac{x}{4}}-\frac{\iota\sqrt{x}}{2}+1}{\sqrt{1-\frac{x}{4}}+\frac{\iota\sqrt{x}}{2}+1}\right)\csc^{-1}\left(\frac{2}{\sqrt{x}}\right)}{\sqrt{x}}.$$

Then for $x=-2$ the following result is deduced

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n2^n}{\binom{2n}{n}(2n+1)^2}&=\sqrt{2}\,\mathrm{Li}_2\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)-\sqrt{2}\,\mathrm{Li}_2\left(\frac{-1+\sqrt{3}}{\sqrt{2}}\right)\\&\quad+\frac{3\sqrt{2}}{2}\zeta(2)-\sqrt{2}\ln\left(\frac{\sqrt{2}-\sqrt{3}+1}{\sqrt{2}+\sqrt{3}-1}\right)\ln\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)\\&=\sqrt{2}\,\mathrm{Li}_2\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)-\sqrt{2}\,\mathrm{Li}_2\left(\frac{-1+\sqrt{3}}{\sqrt{2}}\right)\\&\quad+\frac{3\sqrt{2}}{2}\zeta(2)\end{aligned}$$

$$-\sqrt{2}\ln\left(\frac{\left(1+\sqrt{2}\right)^2-3}{\left(\sqrt{2}+\sqrt{3}\right)^2-1}\right)\ln\left(\sqrt{2-\sqrt{3}}\right)$$

$$=\sqrt{2}\,\mathrm{Li}_2\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)-\sqrt{2}\,\mathrm{Li}_2\left(\frac{-1+\sqrt{3}}{\sqrt{2}}\right)+$$

$$\frac{3\sqrt{2}}{2}\zeta(2)-\frac{\sqrt{2}}{2}\ln\left(\frac{\sqrt{2}}{2+\sqrt{6}}\right)\ln\left(2-\sqrt{3}\right)$$

$$=\sqrt{2}\,\mathrm{Li}_2\left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)-\sqrt{2}\,\mathrm{Li}_2\left(\frac{-1+\sqrt{3}}{\sqrt{2}}\right)+$$

$$\frac{3\sqrt{2}}{2}\zeta(2)-\frac{\sqrt{2}}{2}\ln\left(\sqrt{3}-\sqrt{2}\right)\ln\left(2-\sqrt{3}\right),$$

and the theorem is proved.

□

Theorem 1.9. ***(Alternating version of the Ramanujan's Hypergeometric*** $_3F_2(1)$ ***series).***

The following integral representation for the series in (1.5) *holds:*

$$\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}^2}{2^{4n}(2n+1)}=\frac{2}{\pi}\left(\int_0^1\frac{\ln\left(p+\sqrt{1+p^2}\right)}{p\sqrt{1-p^2}}dp\right).$$

Proof. Invoking the Wallis' well-known integral formula, namely,

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(\sin\theta)^{2n}\,d\theta=\frac{\binom{2n}{n}}{2^{2n}},\qquad n\geq 0.\text{ we have,}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}^2}{2^{4n}(2n+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{2^{2n}(2n+1)}\left(\frac{1}{2^{2n}}\binom{2n}{n}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}}{2^{2n}(2n+1)}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(\sin\theta)^{2n}\,d\theta\right).$$

By the dominated convergence theorem, we are allowed to switch the order of the operators $\sum\limits_{n=0}^{\infty}.$ and $\int\limits_0^{\frac{\pi}{2}}.d\theta$, giving us

$$\sum_{n=0}^{\infty}\frac{(-1)^n\binom{2n}{n}^2}{2^{4n}(2n+1)}=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\left(\sum_{n=0}^{\infty}\frac{\binom{2n}{n}\left(\frac{\iota\sin\theta}{2}\right)^{2n}}{2n+1}\right)d\theta\right]$$

$$= -\frac{2\iota}{\pi}\left(\int_0^{\frac{\pi}{2}} \frac{\sin^{-1}(\iota \sin\theta)}{\sin\theta} d\theta\right)$$
$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \frac{\sinh^{-1}(\sin\theta)}{\sin\theta} d\theta\right).$$

We make the substitution: $\sin\theta = p$, so that $\dfrac{dp}{d\theta} = \cos\theta = \sqrt{1-p^2}$.

It follows that,

$$\frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \frac{\sinh^{-1}(\sin\theta)}{\sin\theta} d\theta\right) = \frac{2}{\pi}\left(\int_0^{1} \frac{\sinh^{-1}(p)}{p\sqrt{1-p^2}} dp\right)$$
$$= \frac{2}{\pi}\left(\int_0^{1} \frac{\ln\left(p+\sqrt{1+p^2}\right)}{p\sqrt{1-p^2}} dp\right).$$

We have to calculate the integral at the RHS, but this seems to be out of reach for the moment. We also note that equivalent expressions of the integral at the RHS have been used a number of times in the Mathematics Stack Exchange online resource.[1]

□

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PERIODICITY OF k-FIBONACCI SEQUENCE MODULO 10^e FOR ODD k

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Abstract. One of the well-known examples of second order linear recurrence relation is the sequence $\{F_n\}$ of Fibonacci numbers. Fibonacci numbers and their several generalizations have interesting properties and various applications in almost every field of science such as in Physics, Biology, Mathematics. One of its interesting properties includes the fact that the sequence $\{F_n\}$ is always periodic when considered modulo any positive integer $m > 1$. There is extensive research which studies the periodicity of various generalizations of $\{F_n\}$ modulo 10^e. The main aim of this article is to consider the periodicity of the sequence $\{F_{(k,n)}\}$ of k-Fibonacci numbers for the odd values of k when considered modulo 10^e; for every integer $e > 1$. The sequence $\{F_{(k,n)}\}$ is one of the interesting generalizations of the sequence $\{F_n\}$, defined by the recurrence relation

$$F_{(k,n)} = kF_{(k,n-1)} + F_{(k,n-2)},\ n \geq 2,$$

where $F_{(k,0)} = 0$, $F_{(k,1)} = 1$. We go through several cases which together covers all the values of odd k and develop the formula for the length of period of $\{F_{(k,n)}\}$ modulo $10^e; e > 1$.

1. Introduction

The Fibonacci sequence $\{F_n\}$ is one of the best known and most studied number sequences. The terms of this sequence are defined by the recurrence

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 1,$$

with the initial terms $F_0 = 0, F_1 = 1$. The Fibonacci sequence has been extended in various manners, with some approaches maintaining the original

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conditions while focusing on retaining the recurrence relation. For widespread research in this topic, one can refer Koshy [17,18], Kramer, Hoggat [9], Livio [11], Vajda [15], Horadam [2], Shannon, Horadam [3], Kilic [7], Yang [16], Ocal, Tuglu, Altinisik [1].

In [14], Falcon and Plaza introduced the k-Fibonacci sequence with one parameter k. The terms of the k-Fibonacci sequence $\{F_{(k,n)}\}$ are defined by the recurrence relation

$$F_{(k,n+1)} = kF_{(k,n)} + F_{(k,n-1)}, \quad n \geq 1,$$

with $F_{(k,0)} = 0, F_{(k,1)} = 1$.

In 1960, Wall [4] studied the periodicity of $\{F_n\}$ extensively. Vinson [10] also studied and extended Wall's work. The periodicity of various generalized Fibonacci sequences modulo an integer $m > 1$ has been studied by several authors. Shah [6] considered the periodicity of the Tribonacci sequence $\{T_n\}$ and proved that

$$T_{n+12\cdot 4\times 10^t} \equiv T_n \pmod{10^t},$$

where $t \geq 1$ and $n \geq 0$. The terms of the sequence $\{T_n\}$ are obtained by the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

with $T_1 = 0$ and $T_2 = T_3 = 1$.

Hathiwala and Shah [8] studied the periodicity of the Tetranacci sequence $\{t_n\}$ defined by

$$t_n = t_{n-1} + t_{n-2} + t_{n-3} + t_{n-4},$$

where $t_1 = t_2 = 0$ and $t_3 = t_4 = 1$. They proved that

$$t_{n+3\cdot 9\times 10^r} \equiv t_n \pmod{10^r},$$

for all $r \geq 4$ and $n \geq 0$.

Patel and Shah [13] studied the length of period of Pell-Lucas sequence $\{PL_n\}$ and proved that

$$PL_{1.2\times 10^e t+n} \equiv PL_n \pmod{10^e},$$

where $e \geq 3$ and $n > 0$. They also introduced the concept of blocks within the period. In [14], Falcon and Plaza considered the periodicity of the k-Fibonacci sequence modulo $m = k^2 + 4$. Koshy [18] studied the periodicity of the Pell sequence $\{P_n\}$, the sequence of Pell numbers.

Dursun Tasci and Gul Ozkan Kizilirmak [5] investigated the period of the k-Fibonacci sequence modulo 2^n. No specific work has been done so far for the periodicity of the sequence $\{F_{(k,n)}\}$ when considered modulo $10^e, e > 1$. In this paper, we consider the periodicity of the sequence $\{F_{(k,n)}\}$ for odd values of k when considered modulo $10^e, e > 1$.

2. The period of $\{F_{(k,n)}\}$

This section is concerned with the basic results related with the periodicity of $\{F_{(k,n)}\}$ for odd values of k, when considered modulo $m > 1$. The results mentioned here are also true for even values of k.

Definition: By $F_{(k,n)} \pmod m$, we mean the sequence of least non-negative residues of the terms of k-Fibonacci sequence taken modulo $m > 1$.

Definition: By $\mu(m)$ (or simply μ), we mean the length of the period of the sequence $\{F_{(k,n)}\}$ for any modulo $m > 1$.

Some of the easy consequences of these definitions are mentioned below.

Lemma 2.1. *The following congruences hold:*

(a) $F_{(k,\mu-2)} \equiv m - k \pmod m$
(b) $F_{(k,\mu-1)} \equiv 1 \pmod m$
(c) $F_{(k,\mu)} \equiv 0 \pmod m$
(d) $F_{(k,\mu+1)} \equiv 1 \pmod m$
(e) $F_{(k,n+\mu r)} \equiv F_{(k,n)} \pmod m$*, for any integer* r.

Marc Renault [12] introduced the periodicity of Fibonacci sequence and also proved some interesting results for periodic nature of $F \pmod m$. The following results can be easily proved using the above lemma and periodic nature of $F_{(k,n)} \pmod m$.

Lemma 2.2. *The sequence* $F_{(k,n)} \pmod m$ *is always periodic.*

Lemma 2.3. *The periodic sequence* $F_{(k,n)} \pmod m$ *repeats from its starting values* $0, 1$.

Fact 2.4. If both $F_{(k,n)} \equiv 0 \pmod m$ and $F_{(k,n+1)} \equiv 1 \pmod m$ hold true, then $\mu(m) \mid n$.

Theorem 2.5. *If* $n \mid m$*, then* $\mu(n) \mid \mu(m)$.

Corollary 2.6. $\mu(m) \mid \mu(mn)$*, for any positive integers* m *and* n.

Theorem 2.7. *If $m = \prod q_i^{e_i}$ is the prime power factorization of a positive integer m, then*

$$\mu(m) = \operatorname{lcm}\left[\mu\left(q_i^{e_i}\right)\right],$$

the least common multiple of $\mu\left(q_i^{e_i}\right)$ for various values of i.

Corollary 2.8. *If m is composite, then $\mu(m)$ is also composite.*

Theorem 2.9. $\mu(\operatorname{lcm}[m,n]) = \operatorname{lcm}[\mu(m), \mu(n)]$.

3. Value of $\mu(2^e)$ for odd values of k

When $k = 1$, clearly $F_{(k,n)} = F_n$, the sequence of Fibonacci numbers, whose periodicity was studied thoroughly in [17]. When $k = 2$, clearly $F_{(k,n)} = P_n$, the sequence of Pell numbers. Results related with $P_n \pmod{2^e}$ can be found in [18]. Throughout this work, we thus consider the periodicity of $\{F_{(k,n)}\}$ for odd integers $k \geq 3$. The following result was proved by Dursun Tasci and Gül Özkan Kizilirmak [5].

Theorem 3.1. *For any integer $e \geq 1$ and when $k \geq 3$ is odd,*

$$\mu(2^e) = 3 \times 2^{e-1}. \tag{3.1}$$

In the next section, we derive the value of $\mu(5^e)$ for the cases $k \equiv \pm 1, \pm 3, 5 \pmod{10}$.

4. Value of $\mu(5^e)$, when $k = 10t \pm 1$

In this section, we first find the value of $\mu(5)$ when $k = 10t \pm 1; t \geq 1$.

Lemma 4.1. $\mu(5) = 20$*, when k is of the form $k = 10t + 1; t \geq 0$.*

Proof. To prove the required result, we need to prove that

$$\begin{aligned} F_{(10t+1,20)} &\equiv F_{(10t+1,0)} \equiv 0 \pmod{5}, \\ \text{and} \quad F_{(10t+1,21)} &\equiv F_{(10t+1,1)} \equiv 1 \pmod{5}. \end{aligned} \tag{4.1}$$

We use induction on t to prove these results. When we consider $t = 0$, we get $F_{(k,n)} = F_n$. It is known that the Fibonacci sequence has periodicity of 20 modulo 5. Thus, $\mu(5) = 20$.
Now, considering $t = 1$, we get $k = 11$. In this case we need to prove that

$$F_{11,20} \equiv 0 \pmod{5} \quad \text{and} \quad F_{11,21} \equiv 1 \pmod{5}.$$

Then the corresponding recurrence relation is

$$\begin{aligned}
F_{11,20} &= 11F_{11,19} + F_{11,18} \\
&= 11\Big[(11)^{18} + 17(11)^{16} + 120(11)^{14} + 455(11)^{12} + 971(11)^{10} \\
&\quad + 1287(11)^{8} + 924(11)^{6} + 330(11)^{4} + 45(11)^{2} + 1\Big] \\
&\quad + \Big[(11)^{17} + 16(11)^{15} + 105(11)^{13} + 364(11)^{11} + 685(11)^{9} \\
&\quad + 792(11)^{7} + 462(11)^{5} + 120(11)^{3} + 9(11)\Big].
\end{aligned}$$

Using Fermat's theorem, we get

$$F_{11,20} \equiv \{1+2+1+2+4+1\}+\{1+1+4+2+2+4\} \equiv 1+4 \equiv 0 \pmod 5.$$

Also, by the corresponding recurrence relation,

$$\begin{aligned}
F_{11,21} &= 11F_{11,20} + F_{11,19} \\
&= 11\Big[(11)^{19} + 18(11)^{17} + 136(11)^{15} + 560(11)^{13} + 1335(11)^{11} \\
&\quad + 1972(11)^{9} + 1716(11)^{7} + 792(11)^{5} + 165(11)^{3} + 10(11)\Big] \\
&\quad + \Big[(11)^{18} + 17(11)^{16} + 120(11)^{14} + 455(11)^{12} + 971(11)^{10} \\
&\quad + 1287(11)^{8} + 924(11)^{6} + 330(11)^{4} + 45(11)^{2} + 1\Big].
\end{aligned}$$

Using Fermat's theorem,

$$F_{11,21} \equiv \{1+3+1+2+1+2\}+\{1+2+1+2+4+1\} \equiv 0+1 \equiv 1 \pmod 5.$$

Therefore, $\mu(5) = 20$ when $k = 10t + 1$. Hence (4.1) holds for $t = 1$.
Now, assume that the result holds for some $t = r$.
That is, let $\mu(5) = 20$ holds for $k = 10r + 1$.

Thus, the following holds:

$$\begin{aligned}
&F_{(10r+1,20)} \equiv 0 \pmod 5, \\
\text{and} \quad &F_{(10r+1,21)} \equiv 1 \pmod 5.
\end{aligned} \tag{4.2}$$

By Lemma 2.1 (b), (c), and (d), we have

$$\begin{aligned}
&(i)\ F_{(10r+1,20)} \equiv 0 \pmod 5, \\
&(ii)\ F_{(10r+1,19)} \equiv 1 \pmod 5, \\
&(iii)\ F_{(10r+1,21)} \equiv 1 \pmod 5.
\end{aligned} \tag{4.3}$$

To prove the result for $t = r + 1$, it is sufficient to show that

$$\begin{aligned} &F_{(10(r+1)+1,20)} \equiv 0 \pmod 5, \\ \text{and} \quad &F_{(10(r+1)+1,21)} \equiv 1 \pmod 5. \end{aligned} \tag{4.4}$$

Now, By the recurrence relation of the k-Fibonacci sequence, for $k = 10(r+1)+1$, we obtain

$$F_{(10r+11,20)} = (10r+11)F_{(10r+11,19)} + F_{(10r+11,18)}. \tag{4.5}$$

Thus, we have

$$\begin{aligned} F_{(10r+11,20)} &= (10r+11)F_{(10r+11,19)} + F_{(10r+11,18)} \\ &= (10r+11)\Big[(10r+11)^{18} + 17(10r+11)^{16} + 120(10r+11)^{14} \\ &\quad + 455(10r+11)^{12} + 971(10r+11)^{10} + 1287(10r+11)^{8} \\ &\quad + 924(10r+11)^{6} + 330(10r+11)^{4} + 45(10r+11)^{2} + 1\Big] \\ &\quad + \Big[(10r+11)^{17} + 16(10r+11)^{15} + 105(10r+11)^{13} \\ &\quad + 364(10r+11)^{11} + 685(10r+11)^{9} + 792(10r+11)^{7} \\ &\quad + 462(10r+11)^{5} + 120(10r+11)^{3} + 9(10r+11)\Big]. \end{aligned}$$

Using Fermat's theorem,

$$F_{(10r+11,20)} \equiv \{1+2+1+2+4+1\} + \{1+1+4+2+2+4\} \equiv 1+4 \equiv 0 \pmod 5.$$

Also, by the recurrence relation of the k-Fibonacci sequence, considering $k = 10(r+1)+1$, we get

$$F_{(10(r+1)+1,21)} = (10(r+1)+1)F_{(10(r+1)+1,20)} + F_{(10(r+1)+1,19)}. \tag{4.6}$$

Thus, we have

$$\begin{aligned}F_{(10r+11,21)} &= (10r+11)F_{(10r+11,20)} + F_{(10r+11,19)}\\ &= (10r+11)\Big[(10r+11)^{19} + 18(10r+11)^{17}\\ &\quad + 136(10r+11)^{15} + 560(10r+11)^{13}\\ &\quad + 1335(10r+11)^{11} + 1972(10r+11)^{9}\\ &\quad + 1716(10r+11)^{7} + 792(10r+11)^{5}\\ &\quad + 165(10r+11)^{3} + 10(10r+11)\Big]\\ &\quad + \Big[(10r+11)^{18} + 17(10r+11)^{16}\\ &\quad + 120(10r+11)^{14} + 455(10r+11)^{12}\\ &\quad + 971(10r+11)^{10} + 1287(10r+11)^{8}\\ &\quad + 924(10r+11)^{6} + 330(10r+11)^{4}\\ &\quad + 45(10r+11)^{2} + 1\Big].\end{aligned}$$

Using Fermat's theorem, we have

$$F_{(10r+11,21)} \equiv \{1+3+1+2+1+2\}+\{1+2+1+2+4+1\} \equiv 0+1 \equiv 1 \pmod 5.$$

Thus, the result holds for $t = r+1$, and hence for all t, as required. □

Lemma 4.2. $\mu(5) = 20$, *when* k *is of the form* $k = 10t-1; t \geq 1$.

This can be proved using the technique similar to the last lemma.

Theorem 4.3. $\mu(5^e) = 20 \times 5^{(e-1)}; e \geq 1$, *when* k *is of the form* $k = 10t+1, t \geq 0$.

Proof. To prove this result, it is first required to prove that

$$\begin{aligned}F_{(k,20\times 5^{(e-1)})} &\equiv F_{(k,0)} \equiv 0 \pmod{5^e},\\ \text{and}\quad F_{(k,20\times 5^{(e-1)}+1)} &\equiv F_{(k,1)} \equiv 1 \pmod{5^e}.\end{aligned} \tag{4.7}$$

From Lemma 4.1 it is clear that the result holds for $e = 1$.
Assume that the result holds for some $e = r > 1$.
Thus,

$$\mu(5^r) = 20 \times 5^{(r-1)}; r > 1. \tag{4.8}$$

Then by lemma 2.1 (b),(c) and (d), we have the following:

$$\begin{aligned}&(i)\quad F_{(k,20\times 5^{r-1})} \equiv 0 \pmod{5^r}\\ &(ii)\quad F_{(k,20\times 5^{r-1}-1)} \equiv 1 \pmod{5^r}\\ &(iii)\quad F_{(k,20\times 5^{r-1}+1)} \equiv 1 \pmod{5^r}.\end{aligned} \tag{4.9}$$

To prove that the result holds for $e = r + 1$ also, it is sufficient to prove that

$$\begin{aligned} F_{(k,20\times 5^r)} &\equiv 0 \pmod{5^{r+1}}, \\ \text{and} \quad F_{(k,20\times 5^r+1)} &\equiv 1 \pmod{5^{r+1}}. \end{aligned} \tag{4.10}$$

By considering $n = 20\times 5^{(r-1)}$ in the recurrence relation $F_{(k,5n)} = kF_{(k,5n-1)} + F_{(k,5n-2)}$, we get

$$F_{(k,20\times 5^r)} = kF_{(k,20\times 5^r-1)} + F_{(k,20\times 5^r-2)}.$$

By lemma 2.1 (a) and (b), we get

$$\begin{aligned} F_{(k,20\times 5^r-1)} &\equiv 1 \pmod{5^{r+1}}, \\ \text{and} \quad F_{(k,20\times 5^r-2)} &\equiv (5^{r+1} - k) \pmod{5^{r+1}}. \end{aligned}$$

Therefore, we get

$$F_{(k,20\times 5^r)} \equiv k \times (1) + (5^{(r+1)} - k) \equiv 0 \pmod{5^{(r+1)}}. \tag{4.11}$$

Also, by considering $n = 20\times 5^{(r-1)}$ in the recurrence relation of k-Fibonacci sequence, since

$$F_{(k,5n+1)} = (k^2 + 1)F_{(k,5n-1)} + kF_{(k,5n-2)},$$

we get

$$F_{(k,20\times 5^r+1)} = (k^2 + 1)F_{(k,20\times 5^r-1)} + kF_{(k,20\times 5^r-2)}.$$

By lemma 2.1 (a) and (b), we get

$$F_{(k,20\times 5^r-1)} \equiv 1 \pmod{5^{r+1}}, \quad F_{(k,20\times 5^r-2)} \equiv (5^{r+1} - k) \pmod{5^{r+1}}.$$

Therefore,

$$F_{(k,20\times 5^r+1)} \equiv (k^2 + 1) \times 1 + k \times (5^{(r+1)} - k) \equiv 1 \pmod{5^{(r+1)}}. \tag{4.12}$$

Then by (4.11),(4.12) and fact 2.3, we have

$$\mu(5^{(r+1)}) \mid 20 \times 5^r. \tag{4.13}$$

Since $5^r \mid 5^{(r+1)}$ implies $\mu(5^r) \mid \mu(5^{(r+1)})$, we get

$$20 \times 5^{(r-1)} \mid \mu(5^{(r+1)}). \tag{4.14}$$

Then by combining (4.13) and (4.14), we get

$$\mu(5^{(r+1)}) = 20 \times 5^{(r-1)} \text{ or } \mu(5^{(r+1)}) = 20 \times 5^r.$$

We shall show that the case $\mu(5^{r+1}) = 20 \times 5^{r-1}$ is not possible. In fact, we show that

$$F_{(k,20\times 5^{(r-1)})} \not\equiv 0 \pmod{5^{(r+1)}}.$$

More precisely, we prove that

$$F_{(k,20\times 5^{(r-1)})} \equiv 5^r R \pmod{5^{(r+1)}}, r \geq 2; \tag{4.15}$$

for some odd R which is not a multiple of 5.
Now, by considering $n = 20\times 5^{(r-2)}$ in the recurrence relation of k-Fibonacci sequence, we get

$$F_{(k,20\times 5^{(r-1)})} = kF_{(k,20\times 5^{(r-1)}-1)} + F_{(k,20\times 5^{(r-1)}-2)}. \tag{4.16}$$

By lemma 2.1 (a) and (b), we have

$$F_{(k,20\times 5^{r-1}-1)} \equiv 1 \pmod{5^r}, \quad F_{(k,20\times 5^{r-1}-2)} \equiv (5^r - k) \pmod{5^r}.$$

In modulo $5^{(r+1)}$, we get

$$\begin{aligned} &F_{(k,20\times 5^{(r-1)}-1)} \equiv 1 \text{ or } (1+5^r), \\ \text{and} \quad &F_{(k,20\times 5^{(r-1)}-2)} \equiv (5^r - k) \text{ or } (2\times 5^r - k). \end{aligned}$$

Then by (4.16) we get

$$F_{(k,20\times 5^{(r-2)})} = k\times(1 \text{ or } 1+5^r)+(5^r-k \text{ or } 2\times 5^r-k) \equiv 5^r R \pmod{5^{(r+1)}}.$$

Therefore, $F_{(k,20\times 5^{r-1})} \not\equiv 0 \pmod{5^{r+1}}$, as R is odd. Thus, we conclude that $\mu(5^{r+1}) = 20\times 5^{r-1}$ is not possible. Hence, $\mu(5^{r+1}) = 20\times 5^r$. □

The following result can be proved using the similar arguments.

Theorem 4.4. *If k is of the form $k = 10t-1$, $t \geq 1$, then $\mu(5^e) = 20\times 5^{e-1}$ for $e \geq 1$.*

5. Value of $\mu(5^e)$, when $k = 10t \pm 3$

In this section, we first find the value of $\mu(5)$ when $k = 10t \pm 3; t \geq 1$.

Theorem 5.1. *If k is of the form $k = 10t+3$, $t \geq 0$, then $\mu(5) = 12$.*

Proof. To prove the required result, we need to prove that

$$\begin{aligned} &F_{(10t+3,12)} \equiv F_{(10t+3,0)} \equiv 0 \pmod{5^e}, \\ \text{and} \quad &F_{(10t+3,13)} \equiv F_{(10t+3,1)} \equiv 1 \pmod{5^e}. \end{aligned} \tag{5.1}$$

If we consider $t = 0$, then we have $k = 3$.
In this case the corresponding recurrence relation is $F_{3,12} = 3\times F_{3,11} + F_{3,10}$.
Thus, we have

$$\begin{aligned} F_{3,12} \equiv 3\times \Big\{&(3)^{10} + 9\times(3)^8 + 28\times(3)^6 + 35\times(3)^4 + 10\times(3)^2 + 1 \\ &+ (3)^9 + 8\times(3)^7 + 21\times(3)^5 + 20\times(3)^3 + 5\times(3)\Big\}. \end{aligned}$$

Using Fermat's theorem, we have

$$\equiv 3 \times \{(3)^2 + 4 \times 1 + 3 \times (3)^2 + 1\} + \{3 + 3 \times (3)^3 + 1 \times 3\} \pmod 5$$
$$F_{3,12} \equiv 3 \times \{4 + 4 + 2 + 1\} + \{3 + 1 + 3\} \pmod 5 \equiv 3 + 2 \equiv 0 \pmod 5.$$

Also, since $F_{3,13} = 3 \times F_{3,12} + F_{3,11}$, we have

$$F_{3,13} = 3 \times F_{3,12} + F_{3,11}$$
$$= 3 \times \Big\{(3)^{11} + 10 \times (3)^9 + 36 \times (3)^7 + 56 \times (3)^5 + 35 \times (3)^3 + 6 \times 3\Big\}$$
$$+ \Big\{(3)^{10} + 9 \times (3)^8 + 28 \times (3)^6 + 35 \times (3)^4 + 10 \times (3)^2 + 1\Big\}.$$

Using Fermat's theorem, we get

$$\equiv 3\Big\{(3)^3 + (3)^3 + 3 + 3\Big\}$$
$$+ \Big\{(3)^2 + 4 + 3(3)^2 + 1\Big\} \pmod 5.$$
$$F_{3,13} \equiv 3 \times \{2 + 2 + 3 + 3\} + \{4 + 4 + 2 + 1\} \pmod 5 \equiv 0 + 1 \equiv 1 \pmod 5.$$

Thus, $\mu(5) = 12$ when $k = 3$.

Now, assume that the result holds for some $t = r \geq 1$.Thus, $\mu(5) = 12$ holds for $k = 10r + 3$.

Thus, by assumption, the following holds:

$$F_{(10r+3,12)} \equiv 0 \pmod 5,$$
$$\text{and } F_{(10r+3,13)} \equiv 1 \pmod 5. \tag{5.2}$$

Then by lemma 2.1 (b),(c) and (d), we have the following:

$$(i)\ F_{(10r+3,12)} \equiv 0 \pmod 5,$$
$$(ii)\ F_{(10r+3,11)} \equiv 1 \pmod 5,$$
$$(iii)\ F_{(10r+3,13)} \equiv 1 \pmod 5. \tag{5.3}$$

To prove that the result holds for $t = r+1$ also, it is sufficient to prove that

$$F_{(10(r+1)+3,12)} \equiv 0 \pmod 5,$$
$$\text{and } F_{(10(r+1)+3,13)} \equiv 1 \pmod 5. \tag{5.4}$$

Now, taking $k = 10(r+1)+3$ in the recurrence relation of sequence $\{F_{(k,n)}\}$, we get

$$F_{(10(r+1)+3,12)} = (10(r+1)+3) \times F_{(10(r+1)+3,11)} + F_{(10(r+1)+3,10)}.$$

Thus, we have

$$\begin{aligned} F_{(10r+13,12)} &= (10r+13)F_{(10r+13,11)} + F_{(10r+13,10)} \\ &= (10r+13)\Big[(10r+13)^{10} + 9(10r+13)^{8} \\ &\quad + 28(10r+13)^{6} + 35(10r+13)^{4} \\ &\quad + 15(10r+13)^{2} + 1\Big] \\ &\quad + \Big[(10r+13)^{9} + 8(10r+13)^{7} \\ &\quad + 21(10r+13)^{5} + 20(10r+13)^{3} \\ &\quad + 5(10r+13)\Big]. \end{aligned}$$

Using Fermat's theorem, we get

$$\equiv 3 \times \{(3)^2 + 4 \times 1 + 3 \times (3)^2 + 1\} + \{3 + 3 \times 2 + 1 \times 3\} (\text{mod } 5)$$

$$F_{(10r+13,12)} \equiv 3 \times \{4+4+2+1\} + \{3+1+3\} (\text{ mod } 5) \equiv 3+2 \equiv 0 (\text{ mod } 5).$$

Also, by considering $k = 10(r+1)+3$ in the recurrence relation of k-Fibonacci sequence, we get

$$F_{(10(r+1)+3,13)} = (10(r+1)+3) \times F_{(10(r+1)+3,12)} + F_{(10(r+1)+3,11)}.$$

Thus, we have

$$\begin{aligned} F_{(10r+13,13)} &= (10r+13)F_{(10r+13,12)} + F_{(10r+13,11)} \\ &= (10r+13)\Big[(10r+13)^{11} + 10(10r+13)^{9} \\ &\quad + 36(10r+13)^{7} + 56(10r+13)^{5} \\ &\quad + 35(10r+13)^{3} + 6(10r+13)\Big] \\ &\quad + \Big[(10r+13)^{10} + 9(10r+13)^{8} \\ &\quad + 28(10r+13)^{6} + 35(10r+13)^{4} \\ &\quad + 10(10r+13)^{2} + 1\Big]. \end{aligned}$$

Using Fermat's theorem, we get

$$\equiv 3 \times \{(3)^3 + 1 \times (3)^3 + 1 \times 3 + 1 \times 3\} + \{(3)^2 + 4 \times 1 + 3 \times (3)^2 + 1\} (\text{mod } 5)$$

$$F_{(10r+13,13)} \equiv 3 \times \{2+2+3+3\} + \{4+4+2+1\} (\text{ mod } 5) \equiv 0+1 \equiv 1 (\text{ mod } 5).$$

Thus, result (5.4) holds for $t = r+1$, and hence for every t, as required. □

The following results can be proved using the similar arguments.

Theorem 5.2. *If k is of the form $k = 10t-3$, $t \geq 1$, then $\mu(5) = 12$.*

Theorem 5.3. *If k is of the form $k = 10t+3$, $t \geq 0$ and $k \equiv -7 \pmod{50}$, then $\mu(5^e) = 12 \times 5^{e-2}$ for all $e > 1$.*

Theorem 5.4. *If k is of the form $k = 10t+3$, $t \geq 0$ and $k \not\equiv -7 \pmod{50}$, then $\mu(5^e) = 12 \times 5^{e-1}$ for all $e > 1$.*

We combine the theorems 5.1,5.3 and 5.4 to write the value of $\mu(5^e)$ when $k = 10t + 3; t \geq 0$ and $e \geq 1$.

Theorem 5.5. *If k is of the form $k = 10t + 3$, $t \geq 0$, then*

$$\mu(5^e) = \begin{cases} 12, & e = 1, \\ 12 \times 5^{e-2}, & e > 1 \text{ and } k \equiv -7 \pmod{50}, \\ 12 \times 5^{e-1}, & e > 1 \text{ and } k \not\equiv -7 \pmod{50}. \end{cases}$$

The following results can be proved on same line for the value of $\mu(5^e)$ when $k = 10t - 3; t \geq 1, e > 1$.

Theorem 5.6. *If k is of the form $k = 10t - 3$, $t \geq 1$, and $k \equiv 7 \pmod{50}$, then $\mu(5^e) = 12 \times 5^{e-2}$ for all $e > 1$.*

Theorem 5.7. *If k is of the form $k = 10t - 3$, $t \geq 1$, and $k \not\equiv 7 \pmod{50}$, then $\mu(5^e) = 12 \times 5^{e-1}$ for all $e \geq 1$.*

We combine the theorems 5.1,5.6 and 5.7 to write the value of $\mu(5^e)$ when $k = 10t - 3; t \geq 1$ and $e \geq 1$.

Theorem 5.8. *If k is of the form $k = 10t - 3$, $t \geq 1$, then*

$$\mu(5^e) = \begin{cases} 12, & e = 1, \\ 12 \times 5^{e-2}, & e > 1 \text{ and } k \equiv 7 \pmod{50}, \\ 12 \times 5^{e-1}, & e > 1 \text{ and } k \not\equiv 7 \pmod{50}. \end{cases}$$

6. Value of $\mu(5^e)$, when $k = 10t + 5$

In this section we consider the case when $k = 10t+5$; $t \geq 0$. It is clear that this can be expressed as $k = 10t + 5 = 5^\alpha(2t' + 1)$, where $\alpha \geq 1$, $t' \geq 0$..

Theorem 6.1. *If $k = 10t + 5$ can be expressed as $k = 5^\alpha(2t' + 1)$, where $\alpha \geq 1$ and $t' \geq 0$, then*

$$\mu(5^e) = \begin{cases} 2, & e < \alpha, \\ 2 \times 5^{e-\alpha}, & e \geq \alpha \text{ and } k = 5^\alpha, \\ 2 \times 5^{e-1}, & e \geq \alpha \text{ and } k \neq 5^\alpha. \end{cases}$$

Proof. We consider $k = 10t + 5 = 5^\alpha(2t' + 1)$, where $\alpha \geq 1, t' \geq 0$.
Case-1: Since $e < \alpha$, we write $\alpha = e + a$ for some $a \geq 1$. Now,

$$\begin{aligned} F_{(5^\alpha(2t'+1),n)} &= 5^\alpha(2t'+1)F_{(5^\alpha(2t'+1),n-1)} + F_{(5^\alpha(2t'+1),n-2)} \\ &= 5^{(e+a)}(2t'+1)F_{(5^\alpha(2t'+1),n-1)} + F_{(5^\alpha(2t'+1),n-2)} \\ &= 5^e \times 5^a(2t'+1)F_{(5^\alpha(2t'+1),n-1)} + F_{(5^\alpha(2t'+1),n-2)} \end{aligned}$$

Therefore,

$$F_{(5^\alpha(2t'+1),n)} \equiv F_{(5^\alpha(2t'+1),n-2)} \pmod{5^e}.$$

Thus, $\mu(5^e) = 2$ when $e < \alpha$.

Case-2: We consider the case $k = 5^\alpha; \alpha \geq 1$ when $\alpha \leq e$. We use induction on e to prove the result. Since $F_{(5^\alpha,n)} = 5^\alpha F_{(5^\alpha,n-1)} + F_{(5^\alpha,n-2)}$, by taking modulo 5, we get $F_{(5^\alpha,n)} \equiv F_{(5^\alpha,n-2)}$. Therefore, $\mu(5) = 2$. Thus, result is true for $e = 1$.

Now, assume that result holds for some $e = r > 1$. Thus,

$$\mu(5^r) = 2 \times 5^{(r-\alpha)}, r > 1. \tag{6.1}$$

Then by lemma 2.1 (b),(c) and (d), we have the following:

$$\begin{aligned} (i) \quad & F_{(k,2\times 5^{r-\alpha})} \equiv 0 \pmod{5^r}, \\ (ii) \quad & F_{(k,2\times 5^{r-\alpha}-1)} \equiv 1 \pmod{5^r}, \\ (iii) \quad & F_{(k,2\times 5^{r-\alpha}+1)} \equiv 1 \pmod{5^r}. \end{aligned} \tag{6.2}$$

To prove that the result holds for $e = r + 1$ also, we need to prove that

$$\begin{aligned} & F_{(k,2\times 5^{r-\alpha+1})} \equiv 0 \pmod{5^{r+1}}, \\ \text{and} \quad & F_{(k,2\times 5^{r-\alpha+1}+1)} \equiv 1 \pmod{5^{r+1}}. \end{aligned} \tag{6.3}$$

Since $F_{(k,5n)} = kF_{(k,5n-1)} + F_{(k,5n-2)}$, by taking $n = 2 \times 5^{(r-\alpha)}$, we get

$$F_{(k,2\times 5^{(r-\alpha+1)})} = kF_{(k,2\times 5^{(r-\alpha+1)}-1)} + F_{(k,2\times 5^{(r-\alpha+1)}-2)}. \tag{6.4}$$

By Lemma 2.1 (a) and (b), we get

$$\begin{aligned} & F_{(k,2\times 5^{r-\alpha+1}-1)} \equiv 1 \pmod{5^{r+1}}, \\ \text{and} \quad & F_{(k,2\times 5^{r-\alpha+1}-2)} \equiv (5^{r+1} - k) \pmod{5^{r+1}}. \end{aligned}$$

Therefore, we get

$$F_{(k,2\times 5^{(r-\alpha+1)})} \equiv k(1) + (5^{r+1} - k) \equiv 0 \pmod{5^{r+1}}. \tag{6.5}$$

Also, by considering $n = 2 \times 5^{(r-\alpha)}$ in the recurrence relation of k-Fibonacci sequence, since

$$F_{(k,5n+1)} = (k^2 + 1)F_{(k,5n-1)} + kF_{(k,5n-2)},$$

we get

$$\begin{aligned} F_{(k,2\times 5^{(r-\alpha+1)}+1)} = {} & (k^2 + 1)F_{(k,2\times 5^{(r-\alpha+1)}-1)} \\ & + kF_{(k,2\times 5^{(r-\alpha+1)}-2)}. \end{aligned}$$

By Lemma 2.1 (a) and (b), we get

$$\begin{aligned} & F_{(k,2\times 5^{r-\alpha+1}-1)} \equiv 1 \pmod{5^{r+1}}, \\ & F_{(k,2\times 5^{r-\alpha+1}-2)} \equiv (5^{r+1} - k) \pmod{5^{r+1}}. \end{aligned}$$

Therefore, we get

$$F_{(k,2\times 5^{(r-\alpha+1)}+1)} \equiv (k^2 + 1) + k(5^{r+1} - k) \equiv 1 \pmod{5^{r+1}}. \tag{6.6}$$

Then by (6.5), (6.6) and Fact 2.3, we have

$$\mu(5^{r+1}) \mid 2 \times 5^{(r-\alpha+1)}. \tag{6.7}$$

Since $5^r \mid 5^{r+1}$ implies $\mu(5^r) \mid \mu(5^{r+1})$, we get

$$2 \times 5^{(r-\alpha)} \mid \mu(5^{r+1}). \tag{6.8}$$

Then by combining (6.7) and (6.8), we have

$$\mu(5^{r+1}) = 2 \times 5^{(r-\alpha)} \quad \text{or} \quad 2 \times 5^{(r-\alpha+1)}.$$

We shall show that the case $\mu(5^{r+1}) = 2 \times 5^{(r-\alpha)}$ is not possible. In fact, we will show that

$$F_{(k,2\times 5^{(r-\alpha)})} \not\equiv 0 \pmod{5^{r+1}}.$$

More precisely, we will prove that

$$F_{(k,2\times 5^{(r-\alpha)})} \equiv 5^r R \pmod{5^{r+1}}, \quad r \geq \alpha \geq 1. \tag{6.9}$$

for some odd R which is not a multiple of 5.

Now, by considering $n = 2 \times 5^{(r-\alpha-1)}$ in the recurrence relation of k-Fibonacci sequence, since

$$F_{(k,5n)} = kF_{(k,5n-1)} + F_{(k,5n-2)},$$

we get

$$F_{(k,2\times 5^{(r-\alpha)})} = kF_{(k,2\times 5^{(r-\alpha)}-1)} + F_{(k,2\times 5^{(r-\alpha)}-2)}. \tag{6.10}$$

By Lemma 2.1 (a) and (b), we have

$$F_{(k,2\times 5^{r-\alpha}-1)} \equiv 1 \pmod{5^r},$$

$$F_{(k,2\times 5^{r-\alpha}-2)} \equiv (5^r - k) \pmod{5^r}.$$

In modulo 5^{r+1}, we get

$$F_{(k,2\times 5^{(r-\alpha)}-1)} \equiv 1 \text{ or } 1 + 5^r,$$

$$F_{(k,2\times 5^{(r-\alpha)}-2)} \equiv (5^r - k) \text{ or } (2 \times 5^r - k).$$

Therefore, by (6.10), we get

$$F_{(k,2\times 5^{(r-\alpha)})} \equiv k(1 \text{ or } 1 + 5^r) + (5^r - k \text{ or } 2 \times 5^r - k) \equiv 5^r R \pmod{5^{r+1}}.$$

Thus, $F_{(k,2\times 5^{(r-\alpha)})} \not\equiv 0 \pmod{5^{r+1}}$, as R is odd. We now conclude that $\mu(5^{r+1}) = 2 \times 5^{(r-\alpha)}$ is not possible. Hence, $\mu(5^{r+1}) = 2 \times 5^{(r-\alpha+1)}$.

Case-3:

Since $\alpha \leq e$, $k = 5^\alpha(2t' + 1)$, when $k \neq 5^\alpha$. To prove this result, it is first required to prove that

$$\begin{aligned} F_{(k,2\times 5^{e-1})} &\equiv F_{(k,0)} \equiv 0 \pmod{5^e}, \\ \text{and} \quad F_{(k,2\times 5^{e-1}+1)} &\equiv F_{(k,1)} \equiv 1 \pmod{5^e}. \end{aligned} \tag{6.11}$$

For $e = 1$, we need to prove that $\mu(5) = 2$. That means to show that

$$F_{(k,2)} \equiv F_{(k,0)} \equiv 0 \pmod 5 \quad \text{and} \quad F_{(k,3)} \equiv F_{(k,1)} \equiv 1 \pmod 5. \tag{6.12}$$

Here we have $k = 5^\alpha(2t' + 1)$.

Considering $\alpha = 1$ and $t' = 0$, we get $k = 5$. Now, by the recurrence relation of k-Fibonacci sequence, we get

$$(i) F_{5,2} = 5 \times F_{(k,1)} + F_{(k,0)} \equiv 5 \times (1) + 0 \equiv 0 (\text{mod } 5).$$

$$(ii) F_{5,3} = 5 \times F_{5,2} + F_{5,1} \equiv 1 (\text{mod } 5^{(r+1)}).$$

Therefore, $\mu(5) = 2$. Thus, result is true for $e = 1$.

Now, assume that result holds for some $e = r \in \mathbb{N}$. Thus,

$$\mu(5^r) = 2 \times 5^{(r-1)}, r > 1. \tag{6.13}$$

Then by lemma 2.1 (b),(c) and (d), we have the following:

$$(i)\ F_{(k,2\times5^{(r-1)})} \equiv 0 \pmod{5^r},$$
$$(ii)\ F_{(k,2\times5^{(r-1)}-1)} \equiv 1 \pmod{5^r},$$
$$(iii)\ F_{(k,2\times5^{(r-1)}+1)} \equiv 1 \pmod{5^r}. \quad (6.14)$$

Now, we prove that the result holds for $e = r+1$ also. That is to prove that

$$F_{(k,2\times5^r)} \equiv 0 (\text{mod } 5^{(r+1)}) \quad \text{and} \quad F_{(k,2\times5^r+1)} \equiv 1 (\text{mod } 5^{(r+1)}). \quad (6.15)$$

By considering $n = 2\times5^r$ in the recurrence relation of k-Fibonacci sequence, we get

$$F_{(k,2\times5^r)} = kF_{(k,2\times5^r-1)} + F_{(k,2\times5^r-2)}. \quad (6.16)$$

By lemma 2.1 (a) and (b), we have

$$F_{(k,2\times5^r-1)} \equiv 1 \pmod{5^{r+1}},$$
$$\text{and} \quad F_{(k,2\times5^r-2)} \equiv (5^{r+1}-k) \pmod{5^{r+1}}.$$

Therefore, we get

$$F_{(k,2\times5^r)} \equiv k\times(1) + (5^{(r+1)}-k) \equiv 0 (\text{mod } 5^{(r+1)}). \quad (6.17)$$

Also, by considering $n = 2\times5^{(r-1)}$ in the recurrence relation of k-Fibonacci sequence, we get

$$F_{(k,2\times5^r+1)} = (k^2+1)F_{(k,2\times5^r-1)} + kF_{(k,2\times5^r-2)}.$$

By lemma 2.1 (a) and (b), we get

$$F_{(k,2\times5^r-1)} \equiv 1 \pmod{5^{r+1}},$$
$$\text{and} \quad F_{(k,2\times5^r-2)} \equiv (5^{r+1}-k) \pmod{5^{r+1}}.$$

Therefore,

$$F_{(k,2\times5^r+1)} \equiv (k^2+1)\times1 + k\times(5^{(r+1)}-k) \equiv 1 (\text{mod } 5^{(r+1)}). \quad (6.18)$$

Then by (6.17),(6.18) and fact 2.3, we have

$$\mu(5^{(r+1)}) \mid 2\times5^r. \quad (6.19)$$

Since $5^r \mid 5^{(r+1)}$ implies $\mu(5^r) \mid \mu(5^{(r+1)})$, we get

$$2\times5^{(r-1)} \mid \mu(5^{(r+1)}). \quad (6.20)$$

Then by combining (6.19) and (6.20), we have

$$\mu(5^{(r+1)}) = 2\times5^{(r-1)} \quad \text{or} \quad \mu(5^{(r+1)}) = 5\times2\times5^{(r-1)} = 2\times5^r.$$

We shall show that the case $\mu(5^{(r+1)}) = 2\times5^{(r-1)}$ is not possible.

In fact, we will show that $F_{(k,2\times5^{(r-1)})} \not\equiv 0(\text{mod } 5^{(r+1)})$. More precisely, we will prove that

$$F_{(k,2\times5^{(r-1)})} \equiv 5^r R(\text{mod } 5^{(r+1)}), r \geq 2; \quad (6.21)$$

for some odd R which is not a multiple of 5.

Now, by considering $n = 2\times5^{(r-2)}$ in the recurrence relation of k-Fibonacci sequence, we get

$$F_{(k,2\times5^{(r-1)})} = kF_{(k,2\times5^{(r-1)}-1)} + F_{(k,2\times5^{(r-1)}-2)}. \quad (6.22)$$

By lemma 2.1 (a) and (b), we have

$$F_{(k,2\times 5^{(r-1)}-1)} \equiv 1 \pmod{5^r}, \quad F_{(k,2\times 5^{(r-1)}-2)} \equiv (5^r - k) \pmod{5^r}.$$

In modulo $5^{(r+1)}$, we get

$$F_{(k,2\times 5^{(r-1)}-1)} \equiv 1 \text{ or } (1+5^r),$$

$$\text{and} \quad F_{(k,2\times 5^{(r-1)}-2)} \equiv (5^r - k) \text{ or } (2\times 5^r - k).$$

Therefore,

$F_{(k,2\times 5^{(r-1)})} = k \times (1 \text{ or } 1+5^r) + (5^r - k \text{ or } 2\times 5^r - k) \equiv 5^r R (\text{mod } 5^{(r+1)})$.

Then, $F_{(k,2\times 5^{(r-1)})} \not\equiv 0 (\text{mod } 5^{(r+1)})$, as R is odd.

Thus, we now conclude that $\mu(5^{(r+1)}) = 2\times 5^{(r-1)}$ is not possible.

Hence, $\mu_F(5^{(r+1)}) = 2\times 5^r$ is true. □

7. Value of $\mu(10^e)$

In this final section, we present a theorem which serves as a conclusion for the entire paper where the value of $\mu(10^e)$ for odd values of k is presented. These results follow from theorem 2.7 and all the results of sections 2 to 6.

Theorem 7.1. *(a) If k is of the form $k = 10t+1$, $t \geq 0$, then*

$$\mu(10^e) = \begin{cases} 60, & e = 1, \\ 300, & e = 2, \\ 15\times 10^{e-1}, & e > 2. \end{cases}$$

(b) If k is of the form $k = 10t-1$, $t \geq 1$, then

$$\mu(10^e) = \begin{cases} 60, & e = 1, \\ 300, & e = 2, \\ 15\times 10^{e-1}, & e > 2. \end{cases}$$

(c) If k is of the form $k = 10t+3$, $t \geq 0$, then

(i) *If $k \equiv -7 \pmod{50}$, then*

$$\mu(10^e) = \begin{cases} 12, & e = 1, 2, \\ 6\times 10^{e-2}, & e > 2. \end{cases}$$

(ii) *If $k \not\equiv -7 \pmod{50}$, then*

$$\mu(10^e) = \begin{cases} 12, & e = 1, \\ 60, & e = 2, \\ 3\times 10^{e-1}, & e > 2. \end{cases}$$

(d) If k is of the form $k = 10t-3$, $t \geq 1$, then

(i) *If $k \equiv 7 \pmod{50}$, then*

$$\mu(10^e) = \begin{cases} 12, & e = 1, 2, \\ 6\times 10^{e-2}, & e > 2. \end{cases}$$

(ii) *If* $k \not\equiv 7 \pmod{50}$, *then*

$$\mu(10^e) = \begin{cases} 12, & e = 1, \\ 60, & e = 2, \\ 3 \times 10^{e-1}, & e > 2. \end{cases}$$

(e) If k *is of the form* $k = 10t + 5 = 5^{\alpha}(2t' + 1)$, $\alpha \geq 1$, *then*

(i) *If* $t' > 0$, $e < \alpha$, *then*

$$\mu(10^e) = \begin{cases} 6, & e = 1, \\ 3 \times 2^{e-1}, & e > 1. \end{cases}$$

(ii) *If* $t' = 0$, $e \geq \alpha$, *then*

$$\mu(10^e) = 3 \times 2^{e-1} \times 5^{e-\alpha}.$$

If $t' \geq 1$, $e \geq \alpha$, *then*

$$\mu(10^e) = \begin{cases} 6, & e = 1, \\ 3 \times 10^{e-1}, & e > 1. \end{cases}$$

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A NOTE ON RIEMANN SOLITONS

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Abstract. In this paper, we discuss the conditions for Killing or parallel vector fields, as well as concircular flatness on the Riemann solitons. Additionally, we analyze totally umbilical, degenerate and totally geodesic hypersurfaces of Riemann solitons using Gauss's and Weingarten's formula respectively.

1. Introduction

An n-dimensional $(n \geq 3)$ differential manifold (M_{main}, g_{rm}) with a non-vanishing vector field V is a Riemann soliton [12] if

$$\frac{1}{2}\mathfrak{L}_V^{ld} g_{rm} \odot g_{rm} + R_{cur.t} = \lambda_c G_{rel.}, \tag{1.1}$$

where g_{rm}, $R_{cur.t}$, λ_c and $\mathfrak{L}_V^{ld}$ are Riemannian metric, curvature tensor of manifold, a real constant and Lie derivative for the vector field V respectively and $\frac{\partial G_{rel.}(t)}{\partial t} = -2R_{cur.t}(g_{rm}(t))$, $G_{rel.} = \frac{1}{2} g_{rm} \odot g_{rm}$, and the Kulkarni-Nomizu product $\odot$ is defined F_1 and F_2 by

$$\begin{aligned}(F_1 \odot F_2)(X,Y,Z,W) = {} & F_1(X,W)F_2(Y,Z) + F_1(Y,Z)F_2(X,W) \\ & - F_1(X,Z)F_2(Y,W) - F_1(Y,W)F_2(X,Z),\end{aligned}$$

for all vector fields X, Y, Z and W on (M_{main}, g_{rm}).
Previous relations and equation (1.1) allow us to express

$$\begin{aligned} & 2R_{cur.t}(X,Y,Z,W) + g_{rm}(X,W)(\mathfrak{L}_V^{ld} g_{rm})(Y,Z) \\ & + g_{rm}(Y,Z)(\mathfrak{L}_V^{ld} g_{rm})(X,W) - g_{rm}(X,Z)(\mathfrak{L}_V^{ld} g_{rm})(Y,W) \\ & - g_{rm}(Y,W)(\mathfrak{L}_V^{ld} g_{rm})(X,Z) = 2\lambda_c[g_{rm}(X,W)g_{rm}(Y,Z) \\ & - g_{rm}(X,Z)g_{rm}(Y,W)]. \end{aligned} \tag{1.2}$$

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Key words and phrases: Riemann soliton, Sectional curvature, Parallel vector field, Killing vector field, Gauss Equation, Weingarten equation.

In recent years, the significance of solitons has been increasingly recognized. Several researchers have contributed to this important area, including Altin and Unal [1], Hamilton [11], Shanmukha and Venkatesh [17], Blaga and Ozgur [7], Barman [2, 3, 4, 5], Duggal and Sharma [8], Erken [10], Nurowski and Randall [13], O'Neill [14], Ozen [15], among others.
The present paper is organized as follows: the introduction provides a brief overview of Riemann solitons. In Section 2, we examine the conditions under which a Riemann soliton admits a Killing vector field or a parallel vector field and is concircularly flat. Finally, we investigate hypersurfaces of Riemann solitons that are totally umbilical, degenerate and totally geodesic,using Gauss's formula and Weingarten formula respectively.

2. Riemann solitons

Theorem 2.1. *If a Riemann soliton possesses a Killing vector field or a parallel vector field, then its sectional curvature is a function of the soliton's real constant.*

Proof. A well-known formula [8] establishes a relationship between the Lie derivative ($\mathfrak{L}^{ld}$) of the metric tensor and the Levi-Civita connection (∇^{lc}), which is given by:

$$(\mathfrak{L}^{ld}_V g_{rm})(X,Y) = g_{rm}(\nabla^{lc}_X V, Y) + g_{rm}(\nabla^{lc}_Y V, X). \tag{2.1}$$

From equations (1.2) and (2.1),

$$\begin{aligned} 2R_{cur.t}(X,Y,Z,W) &+ g_{rm}(X,W)[g_{rm}(\nabla^{lc}_Y V, Z) + g_{rm}(\nabla^{lc}_Z V, Y)] \\ &+ g_{rm}(Y,Z)[g_{rm}(\nabla^{lc}_X V, W) + g_{rm}(\nabla^{lc}_W V, X)] \\ &- g_{rm}(X,Z)[g_{rm}(\nabla^{lc}_Y V, W) + g_{rm}(\nabla^{lc}_W V, Y)] \\ &- g_{rm}(Y,W)[g_{rm}(\nabla^{lc}_X V, Z) + g_{rm}(\nabla^{lc}_Z V, X)] \\ &= 2\lambda_c[g_{rm}(X,W)g_{rm}(Y,Z) - g_{rm}(X,Z)g_{rm}(Y,W)]. \end{aligned} \tag{2.2}$$

Parallel vector field :- [6] A vector field (V) is Parallel vector field, that means, $\nabla^{lc}_X V = 0$, for all X on (M_{main}, g_{rm}).
Killing vector field :- A vector field (V) on (M_{main}, g_{rm}) is a Killing vector field [8] if it satisfies the condition $(\mathfrak{L}^{ld}_V g_{rm})(Y,Z) = 0$, for all Y, Z on (M_{main}, g_{rm}).
Putting $(\mathfrak{L}^{ld}_V g_{rm})(Y,Z) = 0$ in equation (1.2) or $\nabla^{lc}_X V = 0$ in equation (2.2),

for both cases we see

$$R_{cur.t}(X,Y,Z,W) = \lambda_c[g_{rm}(X,W)g_{rm}(Y,Z) - g_{rm}(X,Z)g_{rm}(Y,W)]. \tag{2.3}$$

Sectional curvature :- The sectional curvature $K_{sec.cur.}$ on (M_{main}, g_{rm}) [16] is given by: $K_{sec.cur.} = \frac{R_{cur.t}(X,Y,X,Y)}{[g_{rm}(X,Y)]^2 - g_{rm}(X,X)g_{rm}(Y,Y)}$, for any orthonormal basis X, Y of a tangent plane on (M_{main}, g_{rm}).
Based on the sectional curvature condition and equation (2.3),

$$K_{sec.cur.} = \lambda_c.$$

We have thus shown that the proof is complete. □

Concircular curvature tensor :- Concircular curvature tensor $\mathcal{Z}_{con}$ is defined by ([18], [19])

$$\mathcal{Z}_{con}(X,Y,W,U) = R_{cur.t}(X,Y,W,U)$$
$$-\frac{r}{n(n-1)}[g_{rm}(Y,W)g_{rm}(X,U) - g_{rm}(X,W)g_{rm}(Y,U)], \tag{2.4}$$

where r is the scalar curvature of the manifold.
Concircularly flat :- [19] Concircular curvature tensor $\mathcal{Z}_{con}$ is said to be concircularly flat if and only if $\mathcal{Z}_{con}(X,Y,W,U) = 0$.

Theorem 2.2. *If the Riemann solitons satisfy the conditions of either a Killing vector field or a parallel vector field, then the Riemann solitons will be concircularly flat if and only if the scalar curvature is $n(n-1)\lambda_c$.*

Proof. From the system of equations (2.3) and (2.4),

$$\mathcal{Z}_{con}(X,Y,W,U) = \{\lambda_c - \frac{r}{n(n-1)}\}[g_{rm}(Y,W)g_{rm}(X,U)$$
$$-g_{rm}(X,W)g_{rm}(Y,U)], \tag{2.5}$$

If $\{\lambda_c - \frac{r}{n(n-1)}\} = 0$, that means, $r = n(n-1)\lambda_c$, in equation (2.5) then $\mathcal{Z}_{con}(X,Y,W,U) = 0$ Again if $\mathcal{Z}_{con}(X,Y,W,U) = 0$, then equation (2.5) implies $r = n(n-1)\lambda_c$. Hence, the theorem is proven. □

3. Hypersurface as Riemann solitons

Let us consider an $(n-1)$-dimensional hypersurface $(\tilde{M}_{hyper}, B_{2ff})$ in (M_{main}, g_{rm}), such that $(\tilde{M}_{hyper}, B_{2ff}) \subset (M_{main}, g_{rm})$, where B_{2ff} denotes second fundamental form of hypersurface $(\tilde{M}_{hyper}, B_{2ff})$.

The Gauss formula [8] between Levi-Civita connection $\tilde{\nabla}^{hyper}$ of hypersurface $(\tilde{M}_{hyper}, B_{2ff})$ and Levi-Civita connection ∇^{lc} of manifold (M_{main}, g_{rm}) is defined as

$$\nabla^{lc}_X Y = \nabla^{hyper}_X Y + B_{2ff}(X,Y)N, \tag{3.1}$$

where $X, Y \in (\tilde{M}_{hyper}, B_{2ff})$ and $N \in (M_{main}, g_{rm})$ and $g_{rm}(N,N) = \epsilon_{main} = \pm 1$.
Using equations (2.2) and (3.1),

$$\begin{aligned} 2R_{cur.t}(X,Y,Z,W) &= 2\tilde{R}_{cur.t}(X,Y,Z,W) \\ -g_{rm}(X,W)[B_{2ff}(Y,V)g_{rm}(N,Z) &+ B_{2ff}(Z,V)g_{rm}(N,Y)] \\ -g_{rm}(Y,Z)[B_{2ff}(X,V)g_{rm}(N,W) &+ B_{2ff}(W,V)g_{rm}(N,X)] \\ +g_{rm}(X,Z)[(B_{2ff}(Y,V)g_{rm}(N,W)) &+ B_{2ff}(W,V)g_{rm}(N,Y)] \\ +g_{rm}(Y,W)[g_{rm}(B_{2ff}(X,V)g_{rm}(N,Z) &+ B_{2ff}(Z,V)g_{rm}(N,X)], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} 2\tilde{R}_{cur.t}(X,Y,Z,W) + g_{rm}(X,W)[g_{rm}(\tilde{\nabla}^{hyper}_Y V, Z) + g_{rm}(\tilde{\nabla}^{hyper}_Z V, Y)] \\ +g_{rm}(Y,Z)[g_{rm}(\tilde{\nabla}^{hyper}_X V, W) + g_{rm}(\tilde{\nabla}^{hyper}_W V, X)] \\ -g_{rm}(X,Z)[g_{rm}(\tilde{\nabla}^{hyper}_Y V, W) + g_{rm}(\tilde{\nabla}^{hyper}_W V, Y)] \\ -g_{rm}(Y,W)[g_{rm}(\tilde{\nabla}^{hyper}_X V, Z) + g_{rm}(\tilde{\nabla}^{hyper}_Z V, X)] \\ = 2\lambda_c[g_{rm}(X,W)g_{rm}(Y,Z) - g_{rm}(X,Z)g_{rm}(Y,W)] \end{aligned}$$

is the curvature tensor of the Riemann solitons for hypersurface $(\tilde{M}_{hyper}, B_{2ff})$. Equation (3.2) represents the Riemann soliton equation relating the geometry of the manifold and the hypersurface, derived using the Gauss formula.
Totally umbilical:- [9] $(\tilde{M}_{hyper}, B_{2ff})$ is totally umbilical if $B_{2ff}(X,Y) = kg_{rm}(X,Y), \forall X, Y \in \tilde{M}_{hyper}$, where k is the smooth function.
Using totally umbilical condition on equation (3.2),

$$\begin{aligned} 2R_{cur.t}(X,Y,Z,W) &= 2\tilde{R}_{cur.t}(X,Y,Z,W) \\ -g_{rm}(X,W)[kg_{rm}(Y,V)g_{rm}(N,Z) &+ kg_{rm}(Z,V)g_{rm}(N,Y)] \\ -g_{rm}(Y,Z)[kg_{rm}(X,V)g_{rm}(N,W) &+ kg_{rm}(W,V)g_{rm}(N,X)] \\ +g_{rm}(X,Z)[kg_{rm}(Y,V)g_{rm}(N,W)) &+ kg_{rm}(W,V)g_{rm}(N,Y)] \\ +g_{rm}(Y,W)[kg_{rm}(X,V)g_{rm}(N,Z) &+ kg_{rm}(Z,V)g_{rm}(N,X)], \end{aligned} \tag{3.3}$$

If $R_{cur.t}(X,Y,Z,W)=0$ in equation (3.3),

$$\begin{aligned}2\tilde{R}_{cur.t}(X,Y,Z,W) &= g_{rm}(X,W)[kg_{rm}(Y,V)g_{rm}(N,Z)\\ &+kg_{rm}(Z,V)g_{rm}(N,Y)] + g_{rm}(Y,Z)[kg_{rm}(X,V)g_{rm}(N,W)\\ &+kg_{rm}(W,V)g_{rm}(N,X)] - g_{rm}(X,Z)[kg_{rm}(Y,V)g_{rm}(N,W))\\ &+kg_{rm}(W,V)g_{rm}(N,Y)] - g_{rm}(Y,W)[kg_{rm}(X,V)g_{rm}(N,Z)\\ &+kg_{rm}(Z,V)g_{rm}(N,X)], \end{aligned} \tag{3.4}$$

Hence, the theorem holds:

Theorem 3.1. *If the curvature tensor of the manifold* (M_{main}) *vanishes and the hypersurface is totally umbilical with respect to the Riemann soliton, then by Gauss's formula, the curvature tensor of the hypersurface is given by equation* (3.4).

Degenerate:- [9] A manifold (M_{main}) is degenerate manifold if there exists a vector $N \neq 0$ of M_{main}, such that $g_{rm}(X,N)=0, \forall X \in M_{main}$.
From degenerate condition on manifold and equation (3.2), $R_{cur.t}(X,Y,Z,W) = 2\tilde{R}_{cur.t}(X,Y,Z,W)$, that means, the curvature tensors of manifold (M_{main}) and hypersurface $(\tilde{M}_{hyper})$ are both equal.
Based on the above discussion, the following theorem is presented:

Theorem 3.2. *On degenerate Riemann solitons where the Gauss formula is applicable, the Riemann curvature tensor of the manifold* (M_{main}) *is equal to the Riemann curvature tensor of the hypersurface* $(\tilde{M}_{hyper})$.

The Weingarten formula [9] between Levi-Civita connections $\tilde{\nabla}^{hyper}$ and ∇^{lc} of hypersurface $(\tilde{M}_{hyper}, B_{2ff})$ and manifold (M_{main}, g_{rm}) respectively, is defined as

$$\nabla^{lc}_X N = -A_N X + \tau(X)N, \tag{3.5}$$

where $g_{rm}(B_{2ff}(X,Y),N) = g_{rm}(A_N X,Y)$, $X,Y \in (\tilde{M}_{hyper}, B_{2ff})$; $N \in (M_{main})$ and τ is 1-form of hypersurface.
Substituting $V=N$ into equation (2.2) yields a new equation, which we

then combine with equation (3.5),

$$\begin{aligned}
2R_{cur.t}(X,Y,Z,W) &= 2\lambda_c[g_{rm}(X,W)g_{rm}(Y,Z) \\
&-g_{rm}(X,Z)g_{rm}(Y,W)] + 2g_{rm}(X,W)g_{rm}(B_{2ff}(Y,Z),N) \\
&-g_{rm}(X,W)g_{rm}(Z,N)\tau(Y) - g_{rm}(X,W)g_{rm}(Y,N)\tau(Z) \\
&+2g_{rm}(Y,Z)g_{rm}(B_{2ff}(X,W),N) - g_{rm}(Y,Z)g_{rm}(W,N)\tau(X) \\
&-g_{rm}(Y,Z)g_{rm}(X,N)\tau(W) - 2g_{rm}(X,Z)g_{rm}(B_{2ff}(Y,W),N) \\
&+g_{rm}(X,Z)g_{rm}(W,N)\tau(Y) + g_{rm}(X,Z)g_{rm}(Y,N)\tau(W) \\
&-2g_{rm}(Y,W)g_{rm}(B_{2ff}(X,Z),N) + g_{rm}(Y,W)g_{rm}(Z,N)\tau(X) \\
&+g_{rm}(Y,W)g_{rm}(X,N)\tau(Z). \qquad (3.6)
\end{aligned}$$

Totally geodesic:- [9] $(\tilde{M}_{hyper}, B_{2ff})$ is totally geodesic if $B_{2ff}(X,Y) = 0, \forall X, Y \in \tilde{M}_{hyper}$. Using totally geodesic condition in equation (3.6),

$$\begin{aligned}
2R_{cur.t}(X,Y,Z,W) &= 2\lambda_c[g_{rm}(X,W)g_{rm}(Y,Z) \\
&-g_{rm}(X,Z)g_{rm}(Y,W)] - g_{rm}(X,W)g_{rm}(Z,N)\tau(Y) \\
&-g_{rm}(X,W)g_{rm}(Y,N)\tau(Z) - g_{rm}(Y,Z)g_{rm}(W,N)\tau(X) \\
&-g_{rm}(Y,Z)g_{rm}(X,N)\tau(W) + g_{rm}(X,Z)g_{rm}(W,N)\tau(Y) \\
&+g_{rm}(X,Z)g_{rm}(Y,N)\tau(W) + g_{rm}(Y,W)g_{rm}(Z,N)\tau(X) \\
&+g_{rm}(Y,W)g_{rm}(X,N)\tau(Z). \qquad (3.7)
\end{aligned}$$

We can write the following theorem:

Theorem 3.3. *If the hypersurface* $(\tilde{M}_{hyper})$ *is totally geodesic with respect to the Riemann solitons, then the curvature tensor of the manifolds* (M_{main})*, derived using Weingarten's formula, is given by equation* (3.7).

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AN ELEMENTARY PROOF OF THE CONVERSE OF BROUWER'S FIXED POINT THEOREM

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Abstract. In this paper, we present an elementary proof of the converse of Brouwer's fixed point theorem.

1. Introduction

A topological space X has the fixed point property if every continuous map $f : X \to X$ fixes at least one element of X. There are numerous fixed point theorems in mathematics, the most fundamental among them being Brouwer's fixed point theorem [1], which states that any convex compact subset of the Euclidean space has the fixed point property. On the other hand, as a special case of Theorem 2.3 of [2], it follows that any convex set in the Euclidean Space with the fixed point property is compact. A simpler geometric proof of this classical result in fixed point theory is provided in this paper.

2. Notations

Throughout this paper, C denotes a convex subset of $\mathbb{R}^n$ and $\overline{C}$ denotes its closure.
For $x, y \in \mathbb{R}^n$, we denote
$l(x, y) = \{(1 - \lambda)x + \lambda y; 0 < \lambda < 1\}$
$L(x, y) = \{(1 - \lambda)x + \lambda y; 0 < \lambda\}$

3. Main Results

Lemma 3.1. *If C has the fixed point property, then C is closed.*

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Proof. Let $u \in \overline{C}$. Then $0 \in -u + \overline{C} = \overline{-u + C}$ and $-u + C$ is convex with the fixed point property. Hence, we assume $u = 0$.

Choose a maximal linearly independent subset $S = \{x_1, x_2, \cdots, x_k\} \subseteq C$. By restricting to the subspace spanned by S, we may assume that $k = n$. Then $H = \{a_1x_1 + a_2x_2 + \cdots + a_nx_n;\ a_i > 0\ \forall\ i$ and $a_1 + a_2 + \cdots + a_n < 1\}$ is an open subset of C. Define $y = \dfrac{1}{n+1}(x_1 + x_2 + \cdots + x_n)$. Clearly $y \in H$. Further, $l(0, y) = \{\lambda y; 0 < \lambda < 1\} = \{\dfrac{\lambda}{n+1}(x_1 + x_2 + \cdots + x_n); 0 < \lambda < 1\} \subseteq H \subseteq C$.

Then the map $f : C \to C$ defined by

$$f(x) = \begin{cases} \dfrac{|x|}{2|y|}y & \text{if } |x| \leq |y| \\ \dfrac{1}{2}y & \text{otherwise.} \end{cases}$$

is continuous and hence by our assumption, it has a fixed point. The only possible fixed point of this map is 0 and hence $0 \in C$. Therefore C is closed. □

Lemma 3.2. *If C has the fixed point property, then C is bounded.*

Proof. We may assume $0 \in C$.

Suppose C is unbounded. Then for each $n \in \mathbb{N}$, there exists $x_n \in C$ such that $|x_n| \geq n$. Since C is convex $l(0, x_n) \subseteq C$.

For each $n \in \mathbb{N}$, define $y_n = \dfrac{1}{|x_n|}x_n$.

Since y_n is a sequence on the unit sphere, it has a convergent subsequence (y_{n_k}) that converges, say to y.

Claim: $L(0, y) \subseteq C$

Let $z \in L(0, y)$. Then $\dfrac{z}{|z|} = y$. For each $k \in \mathbb{N}$, define $z_{n_k} = |z|y_{n_k}$.

For any $k \geq |z|$, we have $|x_{n_k}| \geq n_k \geq k \geq |z| = |z_{n_k}|$. Since $l(0, x_{n_k}) \subseteq C$, we have $z_{n_k} \in C$ for all $k \geq |z|$. Thus, we obtain a sequence z_{n_k} in the closed set C, which converges to z. Thus $z \in C$, which proves the claim.

By the above claim, the map $f : C \to C$ defined by $f(x) = (|x| + 1)y$ is a well defined continuous map that does not fix any point, a contradiction to our assumption. Thus C is bounded. □

The above two lemmas, together with the Heine-Borel theorem, establish the converse of Brouwer's fixed point theorem.

Theorem 3.3. *A convex subset of* $\mathbb{R}^n$ *with the fixed point property is compact.*

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CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper we are concerned to prove the existence of solutions of boundary value problem for Caputo-Hadamard fractional differential equations inclusion in a Banach space. Our analysis relies on the set-valued analog of Mönch's fixed point theorem and the technique of measures of noncompactness. Also, we present an example of the main result is included as well as some suggestions for future research.

1. Introduction

The fractional calculus deals with extensions of derivatives and integrals to non-integer orders, which was started to be considered deeply as a powerful tool to reveal the hidden aspects of the dynamics of the complex or hyper complex systems (see [1, 2, 3, 4, 5, 7, 8, 9, 10, 14, 16, 17, 18, 19]). The Hadamard fractional derivative was suggested in early 1892 (see [20]). More recently, a new derivative which involved a Caputo-type modification on the Hadamard derivative known as the Caputo-Hadamard derivative was suggested (see [21]). W. Benhamida, J. R. Graef and S. Hamani (see [22]) studied the boundary value problem

$$ {}^{C}D^{\alpha}y(t) = f(t, y(t)), \quad t \in J = [0, T], \quad 0 < \alpha \leq 1, \tag{1.1}$$

$$ y(T) + y(0) = b\int_0^T y(s)ds, \quad bT \neq 2, \tag{1.2}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, $(\mathbb{E}, \|.\|)$ is a Banach space, $f : J \times \mathbb{E} \to \mathbb{E}$ is a given function and b is a constant. Motivated by the work above, in this paper, we study the existence and uniqueness of positive

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solutions for the m-point boundary value problems for Caputo-Hadamard fractional differential equations of the form

$$({}_{H}^{C}D^{\alpha}y)(t) = h(t), \ \ t \in J = [1, T], \ \ 0 < \alpha \leq 1, \tag{1.3}$$

with fractional boundary condition:

$$y(T) + y(1) = \beta \int_{1}^{T} \sigma(s)ds, \tag{1.4}$$

where ${}_{H}^{C}D^{\alpha}$ is the Caputo-Hadamard fractional derivative of order α,
$f, \sigma : J \times \mathbb{E} \to \mathbb{E}$ are given functions satisfying some assumptions that will b specified later and $\mathbb{E}$ is a Banach space with norm $\|.\|$ and β is a constant. In this work, we present existence results for the problem (1.3)-(1.4) using a method involving a measure of noncompactness and a fixed point theorem of Mönch's type.
This technique was mainly initiated in the monograph of Bana and Goebel (see [23]) and subsequently developed and used in many papers; see, for example, Bana and Sadarangoni (see [24]), Guo et *al.* (see [26]), Lakshmikantham and Leela (see [27]), Mönch's (see [28]), and Szufla (see [29]). The organization of this work is a follows.
In section 2 contains basic definitions and results needed in the sequel. In section 3 is devoted to present the main results describing the existence of solutions for Caputo-Hadamard fractional differential equation (1.3)-(1.4). Finally, we show serval numerical example to explicate our results.

2. Preliminaries

In this section, we present some necessary definitions, lemmas and theorems which will be used in this paper. For more details, we refer to [11, 12, 13, 14, 15, 16, 21, 23, 25, 29].
Let $J = [1, T]$, $T > 0$ and consider the Banach space $C(J, \mathbb{E})$ of continuous functions from J into $\mathbb{E}$ with the norm

$$\|y\|_{\infty} = \sup \left\{ \|y(t)\| : t \in J \right\},$$

Analogously, $C^{n}(J, \mathbb{E})$ is the Banach space of functions $f : J \longrightarrow \mathbb{E}$, where f is n time continuously differentiable on J.

$$\|f\|_{C^n} = \sum_{k=0}^{n} \left\| f^{(k)} \right\|_{C} = \sum_{k=0}^{n} \max \left| f^{(k)}(t) \right|, \quad n \in \mathbb{N}.$$

In particular if $n = 0$, $C^0(J,\mathbb{E}) = C(J,\mathbb{E})$.
Denote by $L^1(J,\mathbb{E})$ be the Banach space of measurable functions $y : J \longrightarrow \mathbb{E}$ which are Bochner integrable with the norm

$$\|y\|_{L^1} = \int_1^T |y(s)|\, ds.$$

Let $L^\infty(J,\mathbb{E})$ be the Banach space of bounded measurable functions $y : J \longrightarrow \mathbb{E}$ equipped with the norm

$$\|y\|_{L^\infty} = \inf\left\{c > 0 : \|y(t)\| \le c,\ a.e\ \ t \in J\right\}.$$

Let $AC(J,\mathbb{E})$ in the Banach space of functions $y : J \longrightarrow \mathbb{E}$, which are absolutely continuous $AC^1(J,\mathbb{E})$ is the space of functions $y : J \longrightarrow \mathbb{E}$ which are absolutely continuous whose first derivative y', is absolutely continuous. Moreover, for a given set V of functions $v : J \longrightarrow \mathbb{E}$ let us denote by

$$V(t) = \left\{v(t) : v \in V\right\},\ t \in J,$$

and

$$V(J) = \left\{v(t) : v \in V,\ t \in J\right\}.$$

For the convenience of the reader, we present some concepts of Caputo-Hadamard type fractional calculus to facilitate the analysis of problem (1.3)-(1.4).

Definition 2.1. (Hadamard fractional integral) [25]
Let $0 < a < b < \infty$ and $y : [a,b] \longrightarrow \mathbb{R}$ the left-sided Hadamard fractional integral of order $\alpha > 0$ of y is defined by

$$_HI^{\alpha}_{a^+}y\,(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s}, \qquad t \in [a,b], \tag{2.1}$$

where Γ stands for the well-known Gamma function by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ and $\log(.) = \log_e(.)$.

Definition 2.2. (Hadamard fractional derivative) [14]
The left-sided Hadamard fractional derivative of order $\alpha \ge 0$ of a continuous function $y : [a,b] \longrightarrow \mathbb{R}$ is given by

$$\begin{aligned} _HD^{\alpha}_{a^+}y\,(t) &= \delta^n I^{n-\alpha}_{a^+}y\,(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} y(s)\frac{ds}{s}, \end{aligned} \tag{2.2}$$

where $n = [\alpha] + 1$ with $[\alpha]$ denotes the integer part of the real number α and $\delta = t\frac{d}{dt}$ provided the right integral converges.

Definition 2.3. (Caputo-Hadamard fractional derivative) [21, 11]
Let $\alpha \geq 0$ and $n = [\alpha] + 1$. If $y(t) \in AC_\delta^n[a, b]$, where $0 < a < b < \infty$ and

$$AC_\delta^n[a,b] = \left\{h : [a,b] \longrightarrow \mathbb{E} : \delta^{n-1}h(t) \in AC\left([a,b], \mathbb{E}\right)\right\}.$$

The Caputo-type modification of the Hadamard fractional derivatives of order α is given by

$${}_H^C D_{a^+}^\alpha y(t) = {}_H D_{a^+}^\alpha \left(y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} (\log \frac{t}{a})^k \right). \tag{2.3}$$

Theorem 2.4. [21]
Let $\alpha \geq 0$ and $n = [\alpha] + 1$. If $y(t) \in AC_\delta^n[a, b]$, where $0 < a < b < \infty$. Then ${}_H^C D_{a^+}^\alpha y(t)$ exist everywhere on $[a, b]$:

(1) *If $\alpha \notin \mathbb{N} - \{0\}$, ${}_H^C D_{a^+}^\alpha y(t)$ can be represented by*

$$\begin{aligned} {}_H^C D_{a^+}^\alpha y(t) &= I_{a^+}^{n-\alpha} \delta^n y(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n y(s) \frac{ds}{s}. \end{aligned} \tag{2.4}$$

(2) *If $\alpha \in \mathbb{N} - \{0\}$, then*

$${}_H^C D_{a^+}^\alpha y(t) = \delta^n y(t), \tag{2.5}$$

in particular

$${}_H^C D_{a^+}^0 y(t) = y(t). \tag{2.6}$$

Caputo-Hadamard fractional derivatives can also be defined on the positive half axis $\mathbb{R}^+$ by replacing a by 0 in formula (2.4) provided that $y(t) \in AC_\delta^n(\mathbb{R}^+)$. Thus one has

$${}_H^C D_{a^+}^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} \delta^n y(s) \frac{ds}{s}. \tag{2.7}$$

Proposition 2.5. [21, 14]
Let $\alpha > 0$, $\beta > 0$, $n = [\alpha] + 1$ and $a > 0$, then

(1) ${}_H I_{a^+}^\alpha \left(\log \frac{t}{a}\right)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{t}{a}\right)^{\beta+\alpha-1}$.

(2) ${}_H^C D_{a^+}^\alpha \left(\log \frac{t}{a}\right)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\log \frac{t}{a}\right)^{\beta-\alpha-1}, \beta > n$.

(3) ${}_H^C D_{a^+}^\alpha \left(\log \frac{t}{a}\right)^k = 0, k = 0, 1, \ldots, n-1$.

Theorem 2.6. [25]
Let $y(t) \in AC_\delta^n[a, b]$, $0 < a < b < \infty$ and $\alpha \geq 0$, $\beta \geq 0$, Then

$${}_H^C D_{a^+}^\alpha \left(I_{a^+}^\beta y\right)(t) = \left(I_{a^+}^{\beta-\alpha} y\right)(t), \tag{2.8}$$

and

$$ {}_{H}^{C}D_{a^+}^{\alpha}\left({}^{C}D_{a^+}^{\beta}y\right)(t) = \left({}^{C}D_{a^+}^{\alpha+\beta}y\right)(t). \tag{2.9} $$

Lemma 2.7. [21] *Let $y \in AC_{\delta}^{n}[1,+\infty)$ and $\alpha > 0$. Then*

$$ {}_{H}I^{\alpha}\left({}_{H}^{C}D^{\alpha}y\right)(t) = y(t) - \sum_{i=0}^{n-1}\frac{\delta^{i}y(1)}{i!}(\log t)^{i}. \tag{2.10} $$

Now let us recall the definition of the Kuratowski measure of noncompactness.

Definition 2.8. [13, 23] Let $\mathbb{E}$ be a Banach space and $\Omega_{\mathbb{E}}$ the bounded subsets of $\mathbb{E}$. The Kuratowski measure of noncompactness is the map $\mu : \Omega_{\mathbb{E}} \longrightarrow [0,\infty]$ defined by

$$ \mu(B) = inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n}B_i, diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_{\mathbb{E}}. $$

The Kuratowski measure of noncompactness satisfies some important properties:

- $\mu(B) = 0 \Leftrightarrow \bar{B}$ is compact (B is relatively compact).
- $\mu(B) = \mu(\bar{B})$.
- $A \subset b \Longrightarrow \mu(A) \leq \mu(B)$.
- $\mu(A+B) \leq \mu(A) + \mu(B)$.
- $\mu(cB) = |c|\,\mu(B); c \in \mathbb{R}$.
- $\mu(convB) = \mu(B)$.

Here $\bar{B}$ and $convB$ denote the closure and the convex hull of the bounded set B, respectively.

Definition 2.9. A multivalued map $f : j\times\mathbb{E} \longrightarrow \mathbb{E}$ is said to be carathéodory if

(1) $t \longrightarrow F(t,u)$ is measurable for each $u \in \mathbb{E}$.
(2) $u \longrightarrow F(t,u)$ is upper semicontinuous for almost all $t \in J$.

Let us now recall Mönch's fixed point theorem and an important lemma.

Theorem 2.10. *(Mönch's fixed point theorem)* [28, 30] *Let D be a bounded, closed, and convex subset of a Banach space $\mathbb{E}$ such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$ V = \overline{conv}N(V) \text{ or } V = N(V)\bigcup\{0\} \text{ implies } \mu(V) = 0, $$

holds for every subset V of D , then N has a fixed point.

Lemma 2.11. [26] *If $V \in C(J,\mathbb{E})$ is a bounded and equicontinuous set, then*

(1) The function $t \longrightarrow \alpha(V(t))$ is continuous on J.
(2) $\alpha\left(\left\{\int_j x(t)dt, x \in V\right\}\right) \leq \int_j \alpha(v(t))dt$.

3. Main Results

Let us start by defining what we meant by a solution of the problem (1.3)-(1.4).

Definition 3.1. A function $y \in AC_\delta^1([1,T];\mathbb{R})$ is said to be a solution of (1.3)-(1.4) if y satisfies the equation ${}_H^C D^\alpha y(t) = f(t, y(t))$ on J and the function y satisfies the condition (1.4).
For the existence of solutions for the problem (1.3)-(1.4).

We need the following auxiliary lemma.

Lemma 3.2. *Let h, $\sigma : [1,+\infty) \longrightarrow \mathbb{R}$ be two continuous functions. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s} \\ & + \frac{\beta}{2} \int_1^T \sigma(s) ds \\ & - \frac{1}{2\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} h(s) \frac{ds}{s}, \end{aligned} \tag{3.1}$$

if and only if y is a solution of the fractional boundary value problem

$$({}_H^C D^\alpha y)(t) = h(t), \ \ t \in J = [1,T], \ \ 0 < \alpha \leq 1, \tag{3.2}$$

$$y(T) + y(1) = \beta \int_1^T \sigma(s) ds, \tag{3.3}$$

Proof. Applying the Hadamard fractional integral of order α to both sides of (1.3) and by using Lemma 2.7, we obtain

$$y(t) = {}_H I^\alpha h(t) + c_1, \tag{3.4}$$

the boundary condition (1.4) implies that

$${}_H I^\alpha h(T) + 2c_1 = \beta \int_1^T \sigma(s) ds,$$

so

$$c_1 = \frac{\beta}{2}\int_1^T \sigma(s)ds - \frac{1}{2}\left[{}_H I^{\alpha} h\left(T\right)\right].$$

Finally, we obtain the solution (3.1)

$$\begin{aligned} y(t) &= \frac{1}{\Gamma\left(\alpha\right)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} h\left(s\right)\frac{ds}{s} \\ &+ \frac{\beta}{2}\int_1^T \sigma(s)ds \\ &- \frac{1}{2\Gamma\left(\alpha\right)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} h\left(s\right)\frac{ds}{s}. \end{aligned}$$

□

3.1. **Existence and uniqueness result via Banach's fixed point theorem.** This section is devoted to the study of the existence of solutions for problem (1.3)-(1.4), in which The function f is defined by $f:[1,T]\times\mathbb{E}\longrightarrow\mathbb{E}$ such that $(\mathbb{E},\|.\|)$ Banach space and $\delta\in\mathbb{E}$. In what follows, we present existence results for the problem (1.3)-(1.4) using a method involving a measure of noncompactness and a fixed point theorem of Mönch type.

In the following, we prove existence results, for the boundary value problem (1.3)-(1.4) by using Mönch fixed point theorem, under the following hypotheses.

$(Hy1)$ The function $f,g:J\times\mathbb{E}\to\mathbb{E}$ is carathéodory conditions,

$(Hy2)$ There exists $p_f,p_g\in L^{\infty}\left(J,\mathbb{R}_+\right)$, such that

$$\|f(t,y)\|\leq p_f(t)\left\|y\right\| \text{ for a.e. } t\in J \text{ and each } y\in\mathbb{E},$$
$$\|g(t,y)\|\leq p_g(t)\left\|y\right\| \text{ for a.e. } t\in J \text{ and each } y\in\mathbb{E}.$$

$(Hy3)$ for a.e. $t\in J$ and each bounded set $B\subset\mathbb{E}$, we have

$$\lim_{k\to 0^+}\alpha\left(f\left(J_{t,k}\times B\right)\right)\leq p_f(t)\alpha(B),$$
$$\lim_{k\to 0^+}\alpha\left(g\left(J_{t,k}\times B\right)\right)\leq p_g(t)\alpha(B),$$

where α is the Kuratowski measure of compactness and $J_{t,k}=[t-k,t]$.

Theorem 3.3. *Assume that hypotheses $(Hy1)$-$(Hy3)$ hold. If*

$$\left[\frac{3\left(\log T\right)^{\alpha}}{2\Gamma\left(\alpha+1\right)}\left\|p_f\right\|_{L^{\infty}}+\frac{|\beta|}{2}\left\|p_g\right\|_{L^{\infty}}(T-1)\right]<1, \tag{3.5}$$

then the boundary value problem (1.3)-(1.4) has at least one solution in $C(J,\mathbb{E})$.

Proof. We transform the problem (1.3)-(1.4) into a fixed point problem by defining an operator

$$N : C(J,\mathbb{E}) \longrightarrow C(J,\mathbb{E}), \tag{3.6}$$

as

$$\begin{aligned} N(y(t)) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s} \\ &+ \frac{\beta}{2} \int_1^T g(s, y(s)) ds \\ &- \frac{1}{2\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s}. \end{aligned} \tag{3.7}$$

Clearly, from Lemma 3.2, the fixed points of N are solutions to the problem (1.3)-(1.4). Let $R > 0$ and consider the set

$$D_R = \{y \in C(J,\mathbb{E}) : \|y\|_\infty \le R\}. \tag{3.8}$$

We can check with out difficulty that, the subset D_R is closed, bounded and convex. We shall show that N satisfies the assumptions of Mönch's fixed point theorem. The proof will be given in three steps.

Step 1: First we prove that N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $C(J,\mathbb{E})$. Then, for each $t \in J$, we have

$$\begin{aligned} |(Ny_n)(t) - (Ny)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^T \frac{1}{s} \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, y_n(s)) - f(s, y(s))| \, ds \\ &+ \frac{|\beta|}{2} \int_1^T |g(s, y_n(s)) - g(s, y(s))| \, ds \\ &+ \frac{1}{2\Gamma(\alpha)} \int_1^T \frac{1}{s} \left(\log \frac{T}{s}\right)^{\alpha-1} |f(s, y_n(s)) - f(s, y(s))| \, ds \\ &\le \frac{3}{2\Gamma(\alpha)} (\log T)^\alpha |f(s, y_n(s)) - f(s, y(s))| \\ &+ \frac{|\beta|(T-1)}{2} |g(s, y_n(s)) - g(s, y(s))|. \end{aligned}$$

Let $\rho > 0$ be such that $\|y_n\|_\infty \le \rho$ and $\|y\|_\infty \le \rho$. Then from $(Hy2)$, we have

$$|f(s, y_n(s)) - f(s, y(s))| \le 2\rho p_f(s) = \sigma_1(s), \tag{3.9}$$

and

$$|g(s, y_n(s)) - g(s, y(s))| \le 2\rho p_g(s) = \sigma_2(s), \tag{3.10}$$

where $\sigma_1 \in L^1(J, \mathbb{R}_+)$ and $\sigma_2 \in L^1(J, \mathbb{R}_+)$.
Since f, g are Carathéodory functions, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Then $N : D_R \longrightarrow D_R$ is consequently continuous on $C(J, \mathbb{E})$.
Step 2: Second, we show that N maps D_R into itself. For any $y \in D_R$, from $(Hy2)$ and (3.5) imply that for each $t \in J$, we have

$$\begin{aligned}
|N(y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, y(s))| \frac{ds}{s} \\
&\quad + \frac{|\beta|}{2} \int_1^T |g(s, y(s))| \, ds + \frac{1}{2\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} |f(s, y(s))| \frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} p_f \|y\| \frac{ds}{s} \\
&\quad + \frac{|\beta|}{2} \int_1^T p_g \|y\| \, ds + \frac{1}{2\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} p_f \|y\| \frac{ds}{s} \\
&\leq \frac{R}{\Gamma(\alpha+1)} (\log T)^\alpha p_f + R\frac{|\beta|}{2} p_g (T-1) + \frac{R}{2\Gamma(\alpha+1)} (T-1) \\
&\leq R \left[\frac{3(\log T)^\alpha}{2\Gamma(\alpha+1)} \|p_f\|_{L^\infty} + \frac{|\beta|}{2} \|p_g\|_{L^\infty} (T-1)\right] \leq R.
\end{aligned}$$

Step 3: N maps bounded sets into equicontinuous sets of $C([1, T], \mathbb{R})$.
We show that $N(D_R)$ is bounded and equicontinuous. In view of step2 it is clear that $N(D_R)$ is bounded. To show the equicontinuity of $N(D_R)$. Let $t_1, t_2 \in J$, $t_1 < t_2$ and $y \in D_R$. Then

$$\begin{aligned}
&|(Ny)(t_2) - (Ny)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1}\right] |f(s, y(s))| \frac{ds}{s} \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} |f(s, y(s))| \frac{ds}{s} \\
&\leq \frac{R}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1}\right] p_f \frac{ds}{s} \\
&\qquad + \frac{R}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} p_f \frac{ds}{s}
\end{aligned}$$

$$\leq \frac{R\|p_f\|}{\Gamma(\alpha+1)}\left[(\log t_2)^\alpha - (\log t_1)^\alpha\right].$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero. Hence $N(D_R) \subset D_R$. Finally we show that the implication holds. Let $V \subset D_R$ such that $V = \overline{conv}(N(V)\bigcup\{0\})$. Since V is bounded and equicontinuous and therefore the function $t \longrightarrow v(t) = \mu(V(t))$ is continuous on J. By assumption $(Hy2)$ and the properties of the measure μ we have for each $t \in J$.

$$\begin{aligned}
v(t) &\leq \mu\left(N(V)(t)\bigcup\{0\}\right) \\
&\leq \mu(N(V)(t)) \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} p_f(s)\mu(V(s))\frac{ds}{s} + \frac{|\beta|}{2}\int_1^T p_g\mu(V(s))ds \\
&+ \frac{1}{2\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1} p_f(s)\mu(V(s))\frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\alpha+1)}(\log T)^\alpha\|p_f\|_{L^\infty} v(s) + \frac{|\beta|}{2}(T-1)\|p_g\|_{L^\infty} v(s) \\
&+ \frac{1}{2\Gamma(\alpha+1)}(\log T)^\alpha\|p_f\|_{L^\infty} v(s) \\
&\leq \|v\|_\infty\left[\frac{3}{2\Gamma(\alpha+1)}(\log T)^\alpha\|p_f\|_{L^\infty} + \frac{|\beta|}{2}(T-1)\|p_g\|_{L^\infty}\right].
\end{aligned}$$

This means that

$$\|v\|_\infty\left[1-\left(\frac{3}{2\Gamma(\alpha)}(\log T)^\alpha\|p_f\|_{L^\infty} + \frac{|\beta|}{2}(T-1)\|p_g\|_{L^\infty}\right)\right] \leq 0; \quad (3.11)$$

Thus

$$\tfrac{3}{2\Gamma(\alpha)}(\log T)^\alpha\|p_f\|_{L^\infty} + \tfrac{|\beta|}{2}(T-1)\|p_g\|_{L^\infty} < 1.$$

□

4. An Example

We give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem. Let

$$E = l^1 = \left\{(y_1, y_2, ..., y_n, ...) : \sum_{n=1}^\infty |y_n| < +\infty\right\},$$

be our Banach space with the norm

$$\|y_n\|_E = \sum_{n=1}^{\infty} |y_n| .$$

Consider the boundary value problem

$$^{C}D^{\alpha}y(t) \in F(t, y(t)), \text{ for almost all } t \in J = [0, T], \ 0 < \alpha \leq 1, \tag{4.1}$$

with fractional boundary condition

$$y(T) + y(0) = b\int_1^T y(s)ds, \ bT \neq 2, \tag{4.2}$$

where $^{C}D^{\alpha}$ is the Caputo fractional derivative, $F : [0,1] \times \mathbb{R} \longrightarrow P(\mathbb{R})$ is a multivalued map, $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. Set

$$F(t, y) = \{v \in \mathbb{R} : f_1(t, y) \leq v \leq f_2(t, y)\},$$

where f_1, $f_2 : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$. We assume that for each $t \in [0,1]]$, $f_1(t, .)$ is lower semi continuous (i.e., the set $\{y \in \mathbb{R} : f_1(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in [0,1]]$, $f_2(t, .)$ is upper semi-continuous (i.e., the set the set $\{y \in \mathbb{R} : f_2(t, y) > \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in C([0,1]; \mathbb{R}^+)$ and $\psi[0;\infty) \longrightarrow (0;\infty)$ continuous and nondecreasing such that

$$\max(|f_1(t, y)|, |f_2(t, y)|) \leq p(t)\psi(|y|), \quad t \in [0,1], \ y \in \mathbb{R}.$$

It is clear that F is compact and convex-valued, and it is upper semi-continuous. Assume there exists a number $M > 0$ such that

$$\left[\frac{3(\log T)^{\alpha}}{2\Gamma(\alpha+1)}\|p_f\|_{L^{\infty}} + \frac{|\beta|}{2}\|p_g\|_{L^{\infty}}(T-1)\right] < 1.$$

Since all the conditions of Theorem 3.3 are satisfied, problem (4.1)-(4.2) has at least one solution y on $[0,1]$.

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ON CLOSED FORMS FOR TWO LOGARITHMIC INTEGRALS

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Abstract. The author aims at deriving two closed form integral expressions for the logarithmic functions by evaluating certain trigonometric integrals and by making use of a *log–sine integral* and a hypergeometric ${}_5F_4\left(\frac{1}{4}\right)$ series result given in this *STUDENT* article.

1. Introduction and Preliminaries

The main focus of the present paper is to establish the following two integral expressions for logarithmic functions, by evaluating certain trigonometric integrals:

$$\int_0^{\frac{\pi}{6}} \ln^3(\sin t)\,dt, \qquad \int_0^{\frac{\pi}{6}} \ln^4(\sin t)\,dt.$$

The works of (among others) Choi *et al.* [6], David Borwein *et al.* [7], Jonathan Borwein and Straub [8], Levrie and Nimbran [10], and Srivastava *et al.* [3] may be cited in connection with the aforementioned developments.

Throughout this paper, $\binom{2k}{k}$ denotes the *central binomial coefficient*, which is defined for $k \geq 1$ by $\binom{2k}{k} = \frac{(2k)!}{(k!)^2}$, $\zeta(z,a)$ denotes the *Hurwitz (or generalized) zeta function*, which is defined by $\zeta(z,a) = \sum_{b=0}^{\infty} \frac{1}{(b+a)^z}$
$(\Re(z) > 1; a \in \mathbb{C} \setminus \{0,-1,-2,\ldots\})$, and $\zeta(z,1) \coloneqq \zeta(z) \coloneqq \sum_{k=1}^{\infty} \frac{1}{k^z}$ is the *Riemann zeta function*, $\mathrm{Cl}_n(\theta)$ denotes the *Clausen function*, which is defined by $\mathrm{Cl}_n(\theta) = \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^n}$, $\quad$ (n is even),

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$\mathrm{Ls}_r(\theta) = -\int_0^\theta \ln^{r-1}\left(2\sin\frac{x}{2}\right)dx \quad (r \in \mathbb{N}\setminus\{1\})$ is the *log–sine integral* of order r and

$(\Gamma_2(z+1))^{-1} = (2\pi)^{z/2} e^{\left(-\frac{z}{2}-\frac{(\gamma+1)z^2}{2}\right)} \prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^k e^{\left(-z+\frac{z^2}{2k}\right)}\right]$ is the *double gamma function*, (cf. [1], [2]; see also [4, pp. 94–96]), where $\gamma = 0.57721...$ is the Euler-Mascheroni constant.

Recall that the double Gamma function can be expressed in terms of the Clausen function (see [5, Eq. (4.5)]):

$$\log\left(\frac{\Gamma_2(1-t)}{\Gamma_2(1+t)}\right) = -t\log\left(\frac{\sin(\pi t)}{\pi}\right) - \frac{1}{2\pi}\mathrm{Cl}_2(2\pi t), \quad 0 < t < 1.$$

In this paper we will use the special case $t = \frac{1}{6}$:

$$\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right) = -\frac{1}{6}\log\left(\frac{1}{2\pi}\right) - \frac{1}{2\pi}\mathrm{Cl}_2\left(\frac{\pi}{3}\right), \quad 0 < t < 1. \tag{1.1}$$

The value of $\mathrm{Cl}_2(\theta)$ for $\theta = \frac{\pi}{3}$ in terms of the Hurwitz zeta function was presented by Choi and Srivastava ([9, Eq.(1.17)]):

$$\mathrm{Cl}_2\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{6}\left[\zeta\left(2,\frac{1}{3}\right) - \frac{2\pi^2}{3}\right]. \tag{1.2}$$

Later on we will also need the value of $\mathrm{Cl}_4\left(\frac{\pi}{3}\right)$. In their article [3, Eq. (3.17), p. 838], H. M. Srivastava, M. L. Glasser and V. S. Adamchik had found the generalized Clausen sum expressible in terms of the Hurwitz zeta function:

$$\mathrm{Cl}_{2k}\left(\frac{\pi}{3}\right) = \sqrt{3}\left[6^{-2k}\left(\zeta\left(2k,\frac{1}{3}\right) + \zeta\left(2k,\frac{1}{6}\right)\right) - \frac{1-3^{-2k}}{2}\zeta(2k)\right], \quad k \in \mathbb{N}. \tag{1.3}$$

Substituting $k = 2$ in both sides of (1.3), we obtain the following equation:

$$\mathrm{Cl}_4\left(\frac{\pi}{3}\right) = \sqrt{3}\left[\frac{1}{1296}\left(\zeta\left(4,\frac{1}{3}\right) + \zeta\left(4,\frac{1}{6}\right)\right) - \frac{40}{81}\zeta(4)\right],$$

Note that from
$$\sum_{b=0}^{\infty}\frac{1}{(6b+1)^4}=\sum_{b=0}^{\infty}\frac{1}{(3b+1)^4}-\frac{1}{16}\sum_{b=0}^{\infty}\frac{1}{(3b+2)^4}$$
and

$$\sum_{b=1}^{\infty}\frac{1}{b^4}=\sum_{b=1}^{\infty}\frac{1}{(3b)^4}+\sum_{b=0}^{\infty}\frac{1}{(3b+1)^4}+\sum_{b=0}^{\infty}\frac{1}{(3b+2)^4}$$
$$\Leftrightarrow\sum_{b=0}^{\infty}\frac{1}{(3b+1)^4}+\sum_{b=0}^{\infty}\frac{1}{(3b+2)^4}=\frac{80}{81}\zeta(4)$$
it immediately follows that, $\zeta\left(4,\frac{1}{6}\right)=17\zeta\left(4,\frac{1}{3}\right)-80\zeta(4)$

and hence

$$\mathrm{Cl}_4\left(\frac{\pi}{3}\right)=\sqrt{3}\left(\frac{1}{72}\zeta\left(4,\frac{1}{3}\right)-\frac{5}{9}\zeta(4)\right). \tag{1.4}$$

2. Main Results

Lemma 2.1. *The following integral formula holds true:*
$$\int_0^{\frac{\pi}{6}}\ln(\sin t)\,dt=-\frac{\pi}{6}\ln(2)-\frac{\sqrt{3}}{12}\left[\zeta\left(2,\frac{1}{3}\right)-\frac{2\pi^2}{3}\right].$$

Proof. In their article [6, Eq. (4.5)], J. Choi, Y. J. Cho and H. M. Srivastava proved that

$$\int_0^{\frac{\pi}{3}}\ln\left(2\sin\frac{x}{2}\right)dx=-\frac{\pi}{3}\ln(2\pi)+2\pi\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right). \tag{2.1}$$

By making the change of variable of $x\mapsto 2t$ for the left side of (2.1) we see that

$$\begin{aligned}\int_0^{\frac{\pi}{3}}\ln\left(2\sin\frac{x}{2}\right)dx&=2\int_0^{\frac{\pi}{6}}\ln(2\sin t)\,dt\\&=2\left(\int_0^{\frac{\pi}{6}}\ln(2)\,dt+\int_0^{\frac{\pi}{6}}\ln(\sin t)\,dt\right)\\&=\frac{\pi}{3}\ln 2+2\int_0^{\frac{\pi}{6}}\ln(\sin t)\,dt\end{aligned}$$

$$= -\frac{\pi}{3}\ln(2\pi) + 2\pi \ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right)$$
$$= -\frac{\pi}{3}\ln(2) - \frac{\pi}{3}\ln(\pi) + 2\pi \ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right)$$

giving us that the following equality holds:

$$\int_0^{\frac{\pi}{6}} \ln(\sin t)\, dt = -\frac{\pi}{6}\ln(\pi) - \frac{\pi}{3}\ln(2) + \pi \ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right).$$

Using (1.1) and (1.2) we get the desired result.

□

Lemma 2.2. *The following integral formula holds true:*

$$\int_0^{\frac{\pi}{6}} \ln^2(\sin t)\, dt = \frac{7}{216}\pi^3 + \frac{\pi}{6}\ln^2 2 + \frac{\sqrt{3}\ln(2)}{6}\left[\zeta\left(2, \frac{1}{3}\right) - \frac{2\pi^2}{3}\right].$$

Proof. In their article [6, Eq. (3.7)], J. Choi, Y. J. Cho and H. M. Srivastava proved that

$$\int_0^{\frac{\pi}{3}} \ln^2\left(2\sin\frac{x}{2}\right) dx = \frac{7}{108}\pi^3. \tag{2.2}$$

Enforcing a substitution of $x = 2t$ on the left side of (2.2) we find

$$\int_0^{\frac{\pi}{3}} \ln^2\left(2\sin\frac{x}{2}\right) dx = 2\int_0^{\frac{\pi}{6}} \ln^2(2\sin t)\, dt$$
$$= 2\left(\int_0^{\frac{\pi}{6}} \ln^2(2)\, dt + \int_0^{\frac{\pi}{6}} \ln^2(\sin t)\, dt + 2\ln 2\int_0^{\frac{\pi}{6}} \ln(\sin t)\, dt\right)$$
$$= \frac{\pi}{3}\ln^2 2 - \frac{2\pi}{3}\ln 2\ln(2\pi) - \frac{2\pi}{3}\ln^2 2 + 4\pi\ln 2\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right)$$
$$+2\int_0^{\frac{\pi}{6}} \ln^2(\sin t)\, dt$$
$$= \frac{7}{108}\pi^3$$

giving us that the following equality holds:

$$\int_0^{\frac{\pi}{6}} \ln^2(\sin t)\, dt = \frac{7}{216}\pi^3 + \frac{\pi}{6}\ln^2 2 + \frac{\pi}{3}\ln 2\ln(2\pi) - 2\pi\ln(2)\ln\left(\frac{\Gamma_2\left(\frac{5}{6}\right)}{\Gamma_2\left(\frac{7}{6}\right)}\right).$$

Using (1.1) and (1.2) we get the desired result.

□

Theorem 2.3. *The following integral formula holds true:*

$$\int_0^{\frac{\pi}{6}} \ln^3(\sin t)\,dt = -\frac{\pi}{6}\ln^3 2 - \frac{\pi}{4}\zeta(3) - \frac{7\pi^3}{72}\ln(2) + \frac{\sqrt{3}\ln^2(2)\,\pi^2}{6}$$
$$-\frac{9\sqrt{3}\ln^2(2)}{4}\sum_{i=0}^{\infty}\frac{1}{(3i+1)^2} - \frac{81\sqrt{3}}{32}\sum_{i=0}^{\infty}\frac{1}{(3i+1)^4}$$
$$+\frac{5\sqrt{3}}{4}\zeta(4).$$

Proof. The integral $\int_0^{\frac{\pi}{6}} \ln^3(2\sin t)\,dt$ reduces to $\int_0^1 \frac{\ln^3(x)}{\sqrt{4-x^2}}dx$ by taking $x = 2\sin t$.

This last integral can be rewritten as a series using the well-known generating function of the central binomial coefficient, $\sum_{n=0}^{\infty}\binom{2n}{n}j^n = \frac{1}{\sqrt{1-4j}}$ and the well-known definite integral formula $\int_0^1 x^{2\alpha}\ln^3(x)\,dx = -\frac{6}{(2\alpha+1)^4}$, $\alpha > -\frac{1}{2}$.

It follows that,

$$\int_0^1 \frac{\ln^3(x)}{\sqrt{4-x^2}}dx = \frac{1}{2}\int_0^1 \ln^3(x)\left[\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{x^2}{16}\right)^n\right]dx$$
$$= \sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{4n}}\left(\frac{1}{2}\int_0^1 x^{2n}\ln^3(x)\,dx\right) = -3\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2^{4n}(2n+1)^4}.$$

In the recent article [10, p. 167], Levrie and Nimbran have evaluated the following series in terms of Clausen's function:

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^4 2^{4n}} := \frac{\pi}{12}\zeta(3) + \frac{3}{4}\operatorname{Cl}_4\left(\frac{\pi}{3}\right), \tag{2.3}$$

and using (1.4), formula (2.3) can be written as follows:

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^4 2^{4n}} = \frac{\pi}{12}\zeta(3) + \frac{3\sqrt{3}}{4}\left(\frac{1}{72}\zeta\left(4,\frac{1}{3}\right) - \frac{5}{9}\zeta(4)\right)$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{6}} \ln^3 (2\sin t)\, dt$$

$$= -\frac{1}{3}\left[\int_0^{\frac{\pi}{6}} \ln^3 2dt + \int_0^{\frac{\pi}{6}} \ln^3 (\sin t)\, dt + 3\ln^2 2\int_0^{\frac{\pi}{6}} \ln (\sin t)\, dt\right]$$

$$-\ln 2\left[\int_0^{\frac{\pi}{6}} \ln^2 (\sin t)\, dt\right]$$

$$= -\frac{\pi}{18}\ln^3 2 - \frac{\sqrt{3}}{12}\ln^2 (2)\left[\zeta\left(2, \frac{1}{3}\right) - \frac{2\pi^2}{3}\right] - \frac{7\pi^3}{216}\ln (2)$$

$$-\frac{1}{3}\int_0^{\frac{\pi}{6}} \ln^3 (\sin t\, dt)\,.$$

The theorem follows immediately.

Here in the proof of Theorem 2.3, Bernstein's theorem [11, Thm. 9.30, p. 243] justifies interchanging the order of integration and summation because of the positivity of the coefficients.

□

Theorem 2.4. *The following integral formula holds true:*

$$\int_0^{\frac{\pi}{6}} \ln^4 (\sin t)\, dt = \frac{\pi}{6}\ln^4 2 + 3\frac{\zeta^2 (3)}{\pi} + \frac{7\pi^3}{36}\ln^2 (2) - \frac{2\sqrt{3}\ln^3 (2)\,\pi^2}{9}$$
$$+3\sqrt{3}\ln^3 (2)\sum_{i=0}^{\infty}\frac{1}{(3i+1)^2} + \frac{81\sqrt{3}\ln 2}{8}\sum_{i=0}^{\infty}\frac{1}{(3i+1)^4}$$
$$-\frac{\sqrt{3}\pi^4 \ln (2)}{18} + \pi\zeta (3)\ln (2) + \frac{2029\pi^5}{60480} + \frac{9}{4\pi}\sum_{k=1}^{\infty}\frac{k^{-6}}{\binom{2k}{k}}.$$

Proof. The log–sine integral $\mathrm{Ls}_5\left(\frac{\pi}{3}\right) = -\int_0^{\frac{\pi}{3}} \ln^4\left(2\sin\frac{x}{2}\right) dx$ reduces to $-2\int_0^{\frac{\pi}{6}} \ln^4 (2\sin t)\, dt$ by taking $x = 2t$.

This last integral can be rewritten as $-2\left[\int_0^{\frac{\pi}{6}} (\ln (2) + \ln (\sin t))^4\, dt\right]$.

Expanding gives,

$$\mathrm{Ls}_5\left(\tfrac{\pi}{3}\right) = -2\int_0^{\frac{\pi}{6}} \ln^4 (2)\, dt - 8\ln^3 (2)\int_0^{\frac{\pi}{6}} \ln (\sin t)\, dt - 2\int_0^{\frac{\pi}{6}} \ln^4 (\sin t)\, dt$$
$$-8\ln (2)\int_0^{\frac{\pi}{6}} \ln^3 (\sin t)\, dt - 12\ln^2 (2)\int_0^{\frac{\pi}{6}} \ln^2 (\sin t)\, dt.$$

Using the results from Lemma 2.1, Lemma 2.2 and Theorem 2.3 the theorem follows, since we have that (see [8, Equations (83f) and (84)], for instance)

$$\mathrm{Ls}_5\left(\frac{\pi}{3}\right) = -\frac{2029}{30240}\pi^5 - \frac{6}{\pi}\zeta^2(3) - \frac{9}{2\pi}\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}k^6}.$$

□

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FOURIER UNIQUENESS (ALMOST) FROM SCRATCH

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Abstract. We give short and direct proofs of the injectivity of the Fourier transforms on the circle and on the real line. Using exactly the same ideas, we also give a new proof of the Stone-Weierstrass Theorem.

1. Introduction

A basic fact in the theory of Fourier series is the "uniqueness theorem", according to which an integrable function $f : [0, 2\pi) \to \mathbb{C}$ is uniquely determined by its Fourier coefficients

$$\widehat{f}(n) := \int_0^{2\pi} f(t)e^{-int}\,\frac{dt}{2\pi}, \quad n \in \mathbb{Z}.$$

Equivalently, if $f : [0, 2\pi) \to \mathbb{C}$ is an integrable function such that $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(t) = 0$ almost everywhere.

However basic, this is a *non-trivial* fact. In spite of that (or, perhaps, because of that), it has become standard, in textbooks on Fourier analysis, to deduce the uniqueness theorem from more general results. For example, one may use the L^1 version of Fejér's Theorem; or, one can first show by any means (Stone-Weierstrass Theorem, continuous version of Fejér's Theorem, ...) that the trigonometric polynomials are dense in the space of continuous 2π-periodic functions, and then conclude by a duality argument.

Now, imagining oneself preparing an introductory course on Fourier series for students knowing the basics of Lebesgue integration but "nothing more" (which may happen for real), it is natural to wonder if the uniqueness theorem can be proved "from scratch", immediately after giving the definition of Fourier coefficients. This is not a merely academic question.

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Indeed, the uniqueness theorem entails the completeness of the trigonometric system in $L^2(0,2\pi)$, which is the key point in the L^2 theory of Fourier series; so it seems quite desirable to have it at one's disposal as quickly as possible.

As it turns out, a direct proof of the uniqueness theorem does exist. This proof goes back at least to Lebesgue (see [7, Chapter II, 24]), and it can be found in the "classics" [10], [1] and [4]. The idea is to first reduce the result to the case of *continuous* functions by observing that if an integrable function $f : [0,2\pi) \to \mathbb{C}$ has all its Fourier coefficients equal to 0, then so does the continuous function $F(t) := c + \int_0^t f(s)\,ds$ for some suitable constant c (since $\widehat{F}(n) = \frac{1}{in}\widehat{f}(n)$ for all $n \neq 0$); and then to prove that a continuous function which is non-zero at some point t cannot have all its Fourier coefficients equal to 0, by using a sequence of trigonometric polynomials peaking at t. The second part of the proof can also be found in some "modern" textbooks, e.g. [9].

In this note, we propose an even more direct proof of the uniqueness theorem. As the one we just outlined, it is completely elementary except for the same non-trivial fact used as a "blackbox", namely that if f is an integrable function on some interval $I \subset \mathbb{R}$ such that $\int_{(a,b)} f(t)\,dt = 0$ for all bounded open intervals $(a,b) \subset I$, then $f(t) = 0$ almost everywhere on I. The latter can be proved for example by applying the so-called *Monotone Class Theorem*, see Section 5.

Essentially the same proof yields the corresponding uniqueness result for the Fourier transform on $\mathbb{R}$; which is of course not surprising. Perhaps more unexpectedly, the very same ideas also lead to a seemingly new proof of the Stone-Weierstrass Theorem.

2. Fourier coefficients

In this section, we prove the uniqueness theorem for Fourier coefficients:

Theorem 2.1. *If $f : [0,2\pi) \to \mathbb{C}$ is an integrable function such that $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(t) = 0$ almost everywhere.*

Proof. It is enough to show that $\int_{(a,b)} f(t)\,dt = 0$ for every open interval $(a,b) \subset [0,2\pi)$. This will follow from the next two facts. Here, by a *trigonometric polynomial*, we mean a linear combination of the functions

e^{int}, $n \in \mathbb{Z}$. Note that, by assumption on f, we have $\int_0^{2\pi} f(t)P(t)\,dt = 0$ for any trigonometric polynomial P.

Fact 2.2. There exists a real-valued trigonometric polynomial q such that $q(t) < 0$ on (a,b) and $q(t) > 0$ on $[0,2\pi)\backslash[a,b]$.

Proof of Fact 2.2. Let

$$q(t) := \sin\left(\frac{t-a}{2}\right)\sin\left(\frac{t-b}{2}\right).$$

Since $\frac{t-a}{2} \in (0,\pi)$ and $\frac{t-b}{2} \in (-\pi,0)$ if $t \in (a,b)$, we have $q(t) < 0$ on (a,b). Similarly, $q(t) > 0$ on $[0,2\pi)\backslash[a,b]$. Moreover, using the well known formula $\sin(u)\sin(v) = \frac{1}{2}\big(\cos(u-v) - \cos(u+v)\big)$, we see that

$$q(t) = \frac{1}{2}\left[\cos\left(\frac{b-a}{2}\right) - \cos\left(t - \frac{a+b}{2}\right)\right],$$

so that q is indeed a trigonometric polynomial. □

Fact 2.3. There exists a sequence of real-valued trigonometric polynomials $(P_k)_{k\in\mathbb{N}}$ with $0 \leqslant P_k \leqslant 1$ for all k, such that $P_k(t) \to 1$ pointwise on (a,b) and $P_k(t) \to 0$ on $[0,2\pi)\backslash[a,b]$.

Proof of Fact 2.3. Let q be the trigonometric polynomial given by Fact 2.2 and let $r(t) := 1 + cq(t)$, where $c > 0$ is chosen in such a way that $0 \leqslant r(t) \leqslant 2$ on $[0,2\pi)$. This is possible since the function q is bounded. Then $0 \leqslant r(t) < 1$ on (a,b) and $1 < r(t) \leqslant 2$ on $[0,2\pi)\backslash[a,b]$. Choose a sequence of positive integers (n_k) such that $n_k > 2^k$ for all k and $2^k/n_k \to 0$, e.g. $n_k = 3^k$, and let

$$P_k(t) := \left(1 - \frac{r(t)^k}{n_k}\right)^{n_k}.$$

Each P_k is a real trigonometric polynomial such that $0 < P_k(t) \leqslant 1$ on $[0,2\pi)$. Moreover, since

$$\log\big(P_k(t)\big) = n_k \log\left(1 - \frac{r(t)^k}{n_k}\right) \sim -r(t)^k,$$

we see that $P_k(t) \to 1$ on (a,b) and $P_k(t) \to 0$ on $[0,2\pi)\backslash[a,b]$. □

It is now easy to show that $\int_{(a,b)} f(t)\,dt = 0$. Let (P_k) be the sequence of trigonometric polynomials given by Fact 2.3. By definition, $P_k(t) \to \mathbf{1}_{(a,b)}(t)$ almost everywhere on $[0,2\pi)$ – in fact, everywhere except at a finite number of points. By the Dominated Convergence Theorem, it follows that $\int_0^{2\pi} P_k(t)f(t)\,dt \to \int_{(a,b)} f(t)\,dt$; which concludes the proof since $\int_0^{2\pi} P_k(t)f(t)\,dt = 0$ for all k. □

Remark 2.1. The above proof can be slightly modified to show that if μ is a complex Borel measure on $\mathbb{T}$ such that $\widehat{\mu}(n) = 0$ for all $n \in \mathbb{Z}$, then $\mu = 0$ (which means, by duality, that the trigonometric polynomials are dense in $\mathcal{C}(\mathbb{T})$). Indeed, considering μ as a measure on $[0, 2\pi)$, the proof shows that $\mu((a, b)) = 0$ for every open interval $(a, b) \subset [0, 2\pi)$ whose endpoints are not atoms of $|\mu|$, the total variation of μ, since in this case the above sequence (P_k) converges $|\mu|$-almost everywhere to $\mathbf{1}_{(a,b)}$. Since $|\mu|$ has only countably many atoms, the family of all such intervals generates the Borel sigma-algebra of $[0, 2\pi)$, and the result follows by the Monotone Class Theorem.

3. Fourier transform on the line

The Fourier transform of an integrable function $f : \mathbb{R} \to \mathbb{C}$ may be defined by

$$\widehat{f}(x) := \int_{\mathbb{R}} f(t)e^{-ixt}dt\,, \quad x \in \mathbb{R}.$$

In this section, we give a direct proof of the uniqueness theorem:

Theorem 3.1. *If $f : \mathbb{R} \to \mathbb{C}$ is an integrable function such that $\widehat{f}(x) = 0$ for all $x \in \mathbb{R}$, then $f(t) = 0$ almost everywhere.*

Proof. We adapt the proof of Theorem 2.1. So, a trigonometric polynomial is now a linear combination of functions of the form $e^{i\lambda t}$, $\lambda \in \mathbb{R}$, and the assumption means that $\int_{\mathbb{R}} f(t)P(t)\,dt = 0$ for every trigonometric polynomial P. It is enough to show that $\int_{(a,b)} f(t)\,dt = 0$ for every bounded open interval $(a, b) \subset \mathbb{R}$.

Let $\varepsilon > 0$. We choose a bounded interval $I \supset (a, b)$ such that $\int_{\mathbb{R}\setminus I} |f| < \varepsilon$.

Having fixed the interval I, there exists a real trigonometric polynomial q such that $q(t) < 0$ on (a, b) and $q(t) > 0$ on $I\setminus[a, b]$. For example, one may take

$$q(t) := \sin\left(\frac{t-a}{T}\right)\sin\left(\frac{t-b}{T}\right),$$

where $T > 0$ is large enough to ensure that $\frac{1}{T}(I - [a, b]) \subset (-\pi, \pi)$.

As above, it follows that there exists a sequence of real trigonometric polynomials (P_k) such that $0 \leqslant P_k \leqslant 1$ on I and $P_k(t) \to \mathbf{1}_{(a,b)}(t)$ almost everywhere on I. Then $\int_I f(t)P_k(t)\,dt \to \int_{(a,b)} f(t)\,dt$ by the Dominated Convergence Theorem. Now, we have $\int_I f(t)P_k(t)\,dt = -\int_{\mathbb{R}\setminus I} f(t)P_k(t)\,dt$ since $\int_{\mathbb{R}} f(t)P_k(t)\,dt = 0$; so $\left|\int_I f(t)P_k(t)\,dt\right| \leqslant \int_{\mathbb{R}\setminus I} |f(t)|\,dt < \varepsilon$ for all $k \in \mathbb{N}$, and hence $\left|\int_{(a,b)} f(t)\right|dt \leqslant \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this concludes the proof. □

Remark 3.2. The above proof is a rather straightforward modification of that of Theorem 2.1. It does not seem immediately clear that Lebesgue's proof outlined in the introduction can be similarly adapted.

Remark 3.3. It is rather tempting to try to prove along the same lines the uniqueness theorem for the Fourier transform on $L^1(G)$, where G is a locally compact abelian group. In this setting, a trigonometric polynomial is a linear combination of *continuous characters* of G, *i.e.* continuous homomorphisms $\gamma : G \to \mathbb{T}$ (considered as complex-valued functions on G). The most obvious analogue of Fact 2.2 would be the following: for any set O in some basis for the topology of G, one can find a real-valued trigonometric polynomial q such that $q < 0$ on O and $q > 0$ on $G\setminus\overline{O}$. However, this already implies that *the continuous characters separate the points of* G; which is a non-trivial result at this level of generality. Even "worse" from our point of view, it seems that the simplest way to prove the separation property is... to use the uniqueness theorem, as for example in [2, Proposition 3.5.2]. So, it appears that our approach cannot yield the uniqueness theorem " from scratch" in this general setting.

4. Stone-Weierstrass

In this section, we recycle the ideas used in the proof of Theorem 2.1 to get a "new proof" of the Stone-Weierstrass Theorem. Compared with the standard one (see e.g. [8]), this proof cannot be considered as elementary since it makes use of the Hahn-Banach Theorem and the Riesz Representation Theorem; yet, we find it rather amusing. As for "novelty", it seems fair to point out that it looks very similar – albeit more self-contained – to the dazzling one-page proof of the lattice version of the Stone-Weierstrass Theorem due to Khurana [6].

Let us recall the statement of the Stone-Weierstrass Theorem.

Theorem 4.1. *Let X be a compact Hausdorff space, and denote by $\mathcal{C}(X)$ the real or complex space of all continuous function on X. If $\mathcal{A}$ is a subalgebra of $\mathcal{C}(X)$ containing $\mathbf{1}$ such that the real-valued functions in $\mathcal{A}$ separate the points of X, then $\mathcal{A}$ is dense in $\mathcal{C}(X)$.*

Note that in the complex case, the standard version of the Stone-Weierstrass Theorem involves a separating algebra closed under complex conjugation. However, this version follows from Theorem 4.1 as stated, since if a linear subspace $\mathcal{A} \subset \mathcal{C}(X)$ is separating and closed under conjugation, then the real functions in $\mathcal{A}$ are also separating. Moreover, it is in fact enough to prove Theorem 4.1 in the real case, for if $\mathcal{A}$ is a linear subspace of $\mathcal{C}(X)$ such that the real functions in $\mathcal{A}$ are dense in $\mathcal{C}_{\mathbb{R}}(X)$, then $\mathcal{A}$ is dense in $\mathcal{C}(X)$.

The proof of (the real version of) Theorem 4.1 relies on the next two facts, which play the roles of Facts 2.2 and 2.3 above. More precisely, Fact 4.3 is a clear analogue of Fact 2.3, whereas Fact 4.2 says that the family of sets satisfying something like the conclusion of Fact 2.2 is rich enough. To state these two facts, we need to fix some notation.

In what follows, we endow the space X with its *Baire sigma-algebra*, *i.e.* the sigma-algebra generated by the compact G_δ sets. The word "measurable" will refer to this sigma-algebra. Note that every $f \in \mathcal{C}(X)$ is measurable, since for any $\alpha \in \mathbb{R}$, the set $\{f \leqslant \alpha\}$ is compact and G_δ.

If $\mathbf{q} = (q_1, \dots, q_d)$ is a finite family of real-valued functions on X, we set

$$O_{\mathbf{q}} := \big\{x \in X : \, q_i(x) < 0 \text{ for all } i \in [\![1, d]\!]\big\}.$$

We also define

$$\breve{O}_{\mathbf{q}} := \big\{x \in X : \, q_i(x) > 0 \text{ for some } i \in [\![1, d]\!]\big\}.$$

If ν is a finite positive Baire measure on X, we will say that a measurable function $q : X \to \mathbb{R}$ is "ν-adequate" if $\nu\big(\{q = 0\}\big) = 0$; and

> we denote by $\mathfrak{O}_\nu$ the family of all subsets of X of the form $O_{\mathbf{q}}$, where $\mathbf{q}$ is a finite family of ν-adequate functions in $\mathcal{A}$.

All sets in $\mathfrak{O}_\nu$ are open and measurable; and by definition of "ν-adequate", we have

$$\nu\big(X \backslash (O_{\mathbf{q}} \cup \breve{O}_{\mathbf{q}})\big) = 0 \quad \text{for every } O_{\mathbf{q}} \in \mathfrak{O}_\nu.$$

Fact 4.2. For any finite positive Baire measure ν on X, the family $\mathfrak{O}_\nu$ is a basis for the topology of X.

Proof. The family $\mathfrak{O}_\nu$ is closed under finite intersections and covers X (since $X = \{-\mathbf{1} < 0\}$); so it is a basis for a topology τ on X, which is coarser than the original topology τ_X since each $O \in \mathfrak{O}_\nu$ is open in X. Since (X, τ_X) is compact, it is enough to show that this topology τ is Hausdorff.

We first observe that for any $q \in \mathcal{A}$, one can find a ν-adequate function $\tilde{q} \in \mathcal{A}$ such that $\|\tilde{q} - q\|_\infty$ is arbitrarily small. Indeed, let $\varepsilon > 0$. For any $\alpha \in (0, \varepsilon)$, the function $q_\alpha := q + \alpha\mathbf{1}$ belongs to $\mathcal{A}$ and $\|q_\alpha - q\|_\infty < \varepsilon$. Moreover, the sets $E_\alpha := \{q_\alpha = 0\}$ are pairwise disjoint. Since ν is a finite measure, it follows that $\nu(E_\alpha) = 0$, *i.e.* q_α is ν-adequate, for all but countably many $\alpha \in (0, \varepsilon)$.

Now, let $x, x' \in X$ with $x \neq x'$. Since $\mathcal{A}$ is separating, one can find a function $f \in \mathcal{A}$ such that $f(x) \neq f(x')$, say $f(x) < f(x')$. Choose α such that $f(x) < \alpha < f(x')$. Then $q := f - \alpha\mathbf{1}$ belongs to $\mathcal{A}$, and $q(x) < 0 < q(x')$. Moreover, by our initial observation, we may also assume that q is ν-adequate. Then $O := \{q < 0\}$ and $O' := \{q > 0\} = \{-q < 0\}$ are τ-open sets separating x and x'. $\square$

Remark. The proof works for any linear subspace $\mathcal{A} \subset \mathcal{C}(X)$ containing $\mathbf{1}$ and separating the points of X.

Fact 4.3. For any finite family $\mathbf{q} = (q_1, \dots, q_d) \subset \mathcal{A}$, one can find a sequence $(P_k)_{k\in\mathbb{N}} \subset \mathcal{A}$ such that $0 \leqslant P_k \leqslant 1$ for all k, $P_k(x) \to 1$ pointwise on $O_\mathbf{q}$ and $P_k(x) \to 0$ pointwise on $\check{O}_\mathbf{q}$.

Proof. The proof of Fact 2.3 shows that for $i = 1, \dots, d$, one can find a sequence $(P_{i,k})_{k\in\mathbb{N}} \subset \mathcal{A}$ such that $0 \leqslant P_{i,k} \leqslant 1$ for all k, $P_{i,k}(x) \to 1$ on $\{q_i < 0\}$ as $k \to \infty$, and $P_{i,k}(x) \to 0$ on $\{q_i > 0\}$. If we set $P_k := \prod_{i=1}^d P_{i,k}$, then the sequence (P_k) does the required job. $\square$

Remark. The proof works for any algebra $\mathcal{A}$ of real-valued bounded functions containing $\mathbf{1}$.

Proof of Theorem 4.1. By the Hahn-Banach Theorem and the "Baire measure version" of the Riesz Representation Theorem (for the latter, see e.g. [8] or the very elegant [3]), it is enough to show that if μ is a real Baire measure on X such that $\int_X P\, d\mu = 0$ for all $P \in \mathcal{A}$, then $\mu = 0$. We fix such a measure μ, and we let $\mathfrak{O} := \mathfrak{O}_{|\mu|}$, where $|\mu|$ is the total variation of μ.

Claim. *The family* $\mathfrak{O}$ *generates the Baire sigma-algebra of* X*, and we have* $\mu(O) = 0$ *for every* $O \in \mathfrak{O}$.

Proof of Claim 4. To prove the first part, it is enough to show that any compact G_δ set $E \subset X$ belongs to the sigma-algebra $\sigma(\mathfrak{O})$ generated by $\mathfrak{O}$. Write $E = \bigcap_{n\in\mathbb{N}} V_n$ where the sets V_n are open in X. Since $\mathfrak{O}$ is a basis for the topology of X by Fact 4.2 and since E is compact, we see that for each $n \in \mathbb{N}$, one can find finitely many sets in $\mathfrak{O}$, say $O_{1,n}, \dots, O_{k_n,n}$, such that $E \subset \bigcup_{k=1}^{k_n} O_{k,n} \subset V_n$. Then $E = \bigcap_{n\in\mathbb{N}} \bigcup_{k=1}^{k_n} O_{k,n}$, so $E \in \sigma(\mathfrak{O})$.

As for the second part, let $O = O_{\mathbf{q}} \in \mathfrak{O}$, where $\mathbf{q} = (q_1, \dots, q_d)$ is a finite family of $|\mu|$-adequate functions in $\mathcal{A}$. Then $|\mu|\big(X\backslash(O_{\mathbf{q}} \cup \check{O}_{\mathbf{q}})\big) = 0$. Hence, the sequence $(P_k) \subset \mathcal{A}$ given by Fact 4.3 converges $|\mu|$-almost everywhere to $\mathbf{1}_{O_{\mathbf{q}}}$. By the Dominated Convergence Theorem, it follows that $\int_X P_k\, d\mu \to \mu(O)$ as $k \to \infty$; which concludes the proof since $\int_X P_k\, d\mu = 0$ for all k. □

Since the family $\mathfrak{O}$ is closed under finite intersections, we can now conclude that $\mu = 0$ by Claim 4 and the Monotone Class Theorem. □

5. The Monotone Class Theorem

In this "appendix", for the sake of completeness, we prove the version of the Monotone Class Theorem we have used in the previous sections.

Let X be an abstract set, and denote by 2^X the family of all subsets of X. We will say that family $\mathfrak{M} \subset 2^X$ is a *monotone class* if it has the following properties:

- $X \in \mathfrak{M}$;
- if $A, A' \in \mathfrak{M}$ and $A \subset A'$, then $A'\backslash A \in \mathfrak{M}$;
- if $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of sets in $\mathfrak{M}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{A}$.

The terminology "monotone class" is not universally used for such families; many people say "Dynkin system" instead.

The usefulness of this notion is due to the fact that there exist natural families of sets which are easily seen to be monotone classes but are *a priori* not sigma-algebras. For example, if $(X, \mathfrak{B})$ is a measurable space and if μ_1, μ_2 are two finite positive measures on $(X, \mathfrak{B})$ – or, more generally, two complex measures – such that $\mu_1(X) = \mu_2(X)$, then the family $\mathfrak{M} = \{A \in \mathfrak{B};\ \mu_1(A) = \mu_2(A)\}$ is a monotone class.

As stated for example in [5], the Monotone Class Theorem reads as follows.

Theorem 5.1. *Let X be an abstract set, and let $\mathfrak{O}$ be a family of subsets which is closed under finite intersections. If $\mathfrak{M} \subset 2^X$ is a monotone class containing $\mathfrak{O}$, then $\mathfrak{M}$ contains the sigma-algebra $\sigma(\mathfrak{O})$ generated by $\mathfrak{O}$.*

Proof. The idea is to show that the monotone class $\mathfrak{M}_0$ generated by $\mathfrak{O}$ is a sigma-algebra. For any $C \in \mathfrak{M}_0$, let $\mathfrak{M}_C := \{E \in 2^X : E \cap C \in \mathfrak{M}_0\}$, and observe that $\mathfrak{M}_C$ is a monotone class. Now, the proof proceeds in three steps, as follows.

(i) For any $A \in \mathfrak{O}$, the monotone class $\mathfrak{M}_A$ contains $\mathfrak{O}$ because $\mathfrak{O}$ is stable under finite intersections. It follows that $A \cap B \in \mathfrak{M}_0$ for any $A \in \mathfrak{O}$ and all $B \in \mathfrak{M}_0$.

(ii) By (i), the monotone class $\mathfrak{M}_B$ contains $\mathfrak{O}$ for any $B \in \mathfrak{M}_0$. It follows that $\mathfrak{M}_0$ is closed under finite intersections.

(iii) Since the monotone class $\mathfrak{M}_0$ is closed under complementation, it is closed under finite unions by (ii). So it is in fact closed under countable unions, and hence it is a sigma-algebra.

□

As an immediate consequence, we get

Corollary 5.2. *Let $(X, \mathfrak{B})$ be a measurable space, and let $\mathfrak{O}$ be a family of measurable sets which is closed under finite intersections. If μ is a complex measure on $(X, \mathfrak{B})$ such that $\mu(O) = 0$ for all $O \in \mathfrak{O}$, then $\mu(A) = 0$ for all $A \in \sigma(\mathfrak{O})$.*

Proof. As observed above, the family $\mathfrak{M} = \{A \in \mathfrak{B};\ \mu(A) = 0\}$ is a monotone class. □

And finally, the "blackbox" result mentioned in the introduction:

Corollary 5.3. *Let I be an interval of $\mathbb{R}$, and let $f : I \to \mathbb{C}$ be a Lebesgue integrable function. If $\int_{(a,b)} f(t)\, dt = 0$ for all bounded open intervals $(a, b) \subset I$, then $f(t) = 0$ almost everywhere.*

Proof. We may assume that the function f is Borel; and considering $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ separately, we may also assume that f is real-valued. By the Monotone Class Theorem and since the family $\mathfrak{O}$ of all bounded open intervals $(a, b) \subset I$ generates the Borel sigma-algebra of I, we see that $\int_A f(t)\, dt = 0$ for every Borel set $A \subset I$. Applying that to the sets $A^+ := \{f > 0\}$ and $A^- := \{f < 0\}$, the result follows. □

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CIRCULANT MATRICES FORM A RING: A PROOF WITHOUT USING GROUP RINGS

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Abstract. Let R be a ring with or without 1 and $\mathrm{C}(n, R)$ be the subset of $\mathrm{M}(n, R)$ of (row) circulant matrices. It is known that $\mathrm{C}(n, R)$ is a subring of $\mathrm{M}(n, R)$. One way to prove this result is using the group ring of a finite group. In this note, we give a simple proof of this fact avoiding the use of group rings and using only elementary linear algebra.

1. Introduction

Let R be a ring (not necessarily commutative) with or without 1 and $\mathrm{C}(n, R)$ be the subset of $\mathrm{M}(n, R)$ of (row) circulant matrices. The question that naturally arises to a curious undergraduate student is that if $\mathrm{C}(n, R)$ is a subring of $\mathrm{M}(n, R)$. In Theorem 1 in [2], an isomorphism is established between the group ring RG of any finite group G of order n and a certain subring $\mathcal{S}$ (explained later) of $\mathrm{M}(n, R)$. By specializing to the case when G is cyclic of order n, the fact that $\mathrm{C}(n, R)$ is a subring of $\mathrm{M}(n, R)$ and is isomorphic to the group ring RG of a finite cyclic group G is obtained. In this paper, we give an elementary and direct proof of the fact that $\mathrm{C}(n, R)$ is a subring of $\mathrm{M}(n, R)$ avoiding the use of group rings and using only basic linear algebra. To be precise, we prove

Theorem 1.1 (Main Theorem). $\mathrm{C}(n, R)$ *is a subring of* $\mathrm{M}(n, R)$.

Our proof is accessible to an undergraduate student with a basic knowledge of rings, matrices and linear maps.

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2. Notations

In this section, we set up some notation and briefly explain how we can show that $\mathrm{C}(n,R)$ is a subring of $\mathrm{M}(n,R)$. Throughout, we let $\mathrm{C}(n,R)$ be the subset of $\mathrm{M}(n,R)$ consisting of matrices of the following type

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & & a_{n-2} \\ \vdots & a_{n-1} & a_0 & \ddots & \vdots \\ a_2 & & \ddots & \ddots & a_1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_0 \end{bmatrix}.$$

We call $\mathrm{C}(n,R)$ the set of (row) *circulant matrices*. Let RG be the group ring of G over R where $G=\{g_1,\ldots,g_n\}$ is a finite group of order n. For $a=\sum_{i=1}^{n}\alpha_{g_i}g_i \in RG$, consider the following matrix

$$M(a)=\begin{bmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \cdots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \cdots & \alpha_{g_2^{-1}g_n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \cdots & \alpha_{g_n^{-1}g_n} \end{bmatrix} \in \mathrm{M}(n,R).$$

Let $\mathcal{S}$ be the subset of $\mathrm{M}(n,R)$ given by

$$\mathcal{S}=\{M(a) \mid a\in RG\}.$$

The main content of Theorem 1 in [2], is to show that $\mathcal{S}$ is a subring of $\mathrm{M}(n,R)$ by using the fact that RG is a ring and establishing an isomorphism between RG and $\mathcal{S}$. Taking G to be a finite cyclic group of order n, it follows that $\mathcal{S}=\mathrm{C}(n,R)$ and hence it is a subring of $\mathrm{M}(n,R)$.

3. MAIN THEOREM

In this section, we give an elementary proof of the main theorem (Theorem 1.1) using only basic linear algebra and avoiding the use of group rings. Before we continue, we prove some preliminary results that we need. For a matrix $X = (x_{ij}) \in \mathrm{M}(n, R)$, we write

$$X = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$$

where X_i is the i^{th} column of the matrix X.

Lemma 3.1. *Let R be a ring with 1 and $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix} \in \mathrm{M}(n, R)$. A is a circulant matrix if and only if $A_k = T^{k-1}A_1$ for all $1 \leq k \leq n$, where*

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} e_2 & e_3 & \cdots & e_n & e_1 \end{bmatrix} \in \mathrm{M}(n, R).$$

Proof. It is easy to see that the ℓ^{th} column of T^{k-1} is given by

$$T_\ell^{k-1} = \begin{cases} e_n, & \text{if } \ell = n - k + 1 \\ e_{(\ell+k-1) \bmod n}, & \text{if } \ell \neq n - k + 1 \end{cases}.$$

Thus we see that

$$T^{k-1} = \begin{bmatrix} e_k & e_{k+1} & \dots & e_n & e_1 & \dots & e_{k-1} \end{bmatrix} \in \mathrm{M}(n, R).$$

Let A be the circulant matrix given by

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & & a_{n-2} \\ \vdots & a_{n-1} & a_0 & \ddots & \vdots \\ a_2 & & \ddots & \ddots & a_1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_0 \end{bmatrix}$$

let $A^\top$ be its transpose. Then for $1 \leq k \leq n$ we have

$$A_k^\top = \begin{bmatrix} a_{k-1} \cdots a_1 & a_0 & a_{n-1} \cdots a_k \end{bmatrix}.$$

Thus

$$T^{k-1}A_1 = e_k a_0 + e_{k+1}a_{n-1} + \cdots + e_n a_k + e_1 a_{k-1} + \cdots + e_{k-1}a_1 = A_k.$$

Conversely, if $T^{k-1}A_1 = A_k$ for all $1 \leq k \leq n$, clearly A is circulant. □

Lemma 3.2. *Let R be a ring with 1 and let $T \in \mathrm{M}(n, R)$ be defined as*

$$T = \begin{bmatrix} e_2 & e_3 & \cdots & e_n & e_1 \end{bmatrix}.$$

Then T is invertible and we have

$$T^{-1} = \begin{bmatrix} e_n & e_1 & \cdots & e_{n-2} & e_{n-1} \end{bmatrix}.$$

Proof. Since T is row equivalent to the identity matrix, it is invertible. A simple computation shows that $T^{-1}T = TT^{-1} = 1$. Indeed, we have

$$T^{-1}T = \begin{bmatrix} T^{-1}e_2 & T^{-1}e_3 & \cdots & T^{-1}e_n & T^{-1}e_1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_{n-1} & e_n \end{bmatrix}$$

and

$$TT^{-1} = \begin{bmatrix} Te_n & Te_1 & \cdots & Te_{n-2} & Te_{n-1} \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_{n-1} & e_n \end{bmatrix}.$$

□

Lemma 3.3. *Let $T \in \mathrm{M}(n, R)$ be as before and $A \in \mathrm{C}(n, R)$. Then we have*

$$AT = TA.$$

Proof. It is enough to show that $T^{-1}AT = A$. For $1 \leq k \leq n$, we have $T_k = e_{(k+1) \bmod n}$. Thus

$$(T^{-1}AT)_k = T^{-1}(AT)_k = T^{-1}AT_k = T^{-1}A_{k+1} = T^{-1}T^k A_1 = T^{k-1}A_1 = A_k.$$

□

3.1. **Proof of Theorem 1.1.** Suppose that R is a ring with 1. Let $A, B \in \mathrm{C}(n, R)$. Suppose that $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$ and $B = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}$. We will show that $A + B, AB \in \mathrm{C}(n, R)$. The other axioms for a ring are easy to check. For $1 \leq k \leq n$, we have

$$(A + B)_k = A_k + B_k = T^{k-1}A_1 + T^{k-1}B_1 = T^{k-1}(A + B)_1.$$

Thus $A + B \in \mathrm{C}(n, R)$. Using Lemma 3.1, it is enough to show that

$$(AB)_k = T^{k-1}(AB)_1.$$

Using Lemma 3.3 we have

$$(AB)_k = AB_k = AT^{k-1}B_1 = T^{k-1}AB_1 = T^{k-1}(AB)_1.$$

It follows that $\mathrm{C}(n, R)$ is a subring of $\mathrm{M}(n, R)$. In the case when R is a ring without 1, we can embed R into a ring $\widetilde{R}$ with 1 (see Theorem 1.10 in [1]) and we get an embedding of $\mathrm{M}(n, R)$ into $\mathrm{M}(n, \widetilde{R})$. The result now follows from the observation that $\mathrm{C}(n, R) = \mathrm{C}(n, \widetilde{R}) \cap \mathrm{M}(n, R)$.

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A SHORT ELEMENTARY PROOF OF THE SANDHAM–AU-YEUNG SERIES

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Abstract. We aim to give a short and direct proof of an old and well-known identity $\sum_{n=1}^{\infty} H_n^2/n^2 = 17\pi^4/360$. The crux of the argument lies in writing the denominator term n of the series as an integral representation. Although the proof is completely elementary, it is based on a less trivial result of Wang and Lyu [18] on the ordinary generating function for the sequence $\{H_n^2/n\}_{n\geq 1}$ and an entry of a generalized integral identity found in Table 110 of the reference book *Nouvelles tables d'intégrales définies* [20].

1. Introduction

Ever since the time of Euler, the so-called nonlinear harmonic sums had been investigated. We mention that a nonlinear harmonic sum is a series which involves products of at least two (generalized) harmonic numbers [17].

We give a truly elementary proof of the identity

$$\left(\frac{H_1}{1}\right)^2 + \left(\frac{H_2}{2}\right)^2 + \left(\frac{H_3}{3}\right)^2 + \cdots = \frac{17}{360}\,\pi^4, \tag{1.1}$$

where $H_n = \sum_{j=1}^{n} j^{-1}$ is the well studied *classical harmonic number*.

The Sandham–Au-Yeung series refers to the problem of evaluating the infinite series in equation (1.1), and it appeared as problem 4305 in *The American Mathematical Monthly* [1, p. 431] proposed by H. F. Sandham and a solution due to Martin Kneser appeared in *The American Mathematical Monthly* [2, p. 267]. There is an interesting historical account in [21].

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There are many short proofs of (1.1), but most rely on additional knowledge. A nice collection is given in [13]. One proof commonly used is based on double integrals and the calculation of the logarithmic integrals with the tail of the dilogarithmic function (see [5, Proposition. A.1] or [6, 7, 16]). A second approach is based on the Abel's summation formula (see [8] or [11]). Other proofs rely on Parseval's theorem [3] or application of residue calculus to ψ expansions [4, p. 24]. Yet other proofs involve a Master Theorem of Series, such as the one in [9, Lemma 4] or calculation of the nonlinear harmonic sums [?]. Without attempting to provide a complete list, there are proofs in [10, 13–15] and references therein.

2. Definitions

Let us first introduce the *classical Bernoulli numbers* B_m defined by

$$B_0 = 1 \quad \text{and} \quad \sum_{i=0}^{m} \binom{m+1}{i} B_i = 0 \quad (m \in \mathbb{N}).$$

For convenience, we recall the definition of the well-known *Polylogarithm function*:

$$\mathrm{Li_p}\,(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^p} \quad (|z| \leq 1; \quad p \in \mathbb{N} \setminus \{1\}).$$

One has in particular

$$\mathrm{Li_p}\,(1) = \zeta(p), \quad \mathrm{Li_2}\,(1) = \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}$$

where ζ is the famous *Riemann Zeta function* $\zeta(p) = \sum_{k=1}^{\infty} k^{-p}$.

We begin with two lemmas that will be needed for the proof of the theorem. The second part of second lemma is a special case of Equation (3.93) in [15].

Lemma 2.1. *The following identity holds:*

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} x^{n-1} = \frac{\mathrm{Li_3}\,(x)}{x} - \frac{\mathrm{Li_2}\,(x) \ln(1-x)}{x} - \frac{1}{3} \left(\frac{\ln^3(1-x)}{x} \right).$$

Proof. Starting with the generating function for the sequence $\{H_n^2/n\}_{n \geq 1}$ given in (see [18, Equation (4.6)] or [13, Equation (10)], for instance)

$$\sum_{n=1}^{\infty} \frac{H_n^2 x^n}{n} = \mathrm{Li_3}\,(x) - \mathrm{Li_2}\,(x) \ln(1-x) - \frac{1}{3} \ln^3(1-x) \quad (|x| \leq 1, \quad x \neq 1). \tag{2.1}$$

Dividing both sides of (2.1) by x, the result immediately follows.

□

Lemma 2.2. *The following equalities hold:*

(a) $\displaystyle\int_0^1 \frac{\mathrm{Li}_3(x)}{x}dx = \frac{1}{90}\pi^4$; *(b)* $\displaystyle\int_0^1 \frac{\mathrm{Li}_2(x)\ln(1-x)}{x}dx = -\frac{1}{72}\pi^4$.

Proof. (a) Observe that the integrand $\frac{\mathrm{Li}_3(x)}{x}$ has a removable singularity at $x = 0$ and as x goes to 0, the integrand goes to 1.

$$\int_0^1 \frac{\mathrm{Li}_3(x)}{x}dx = \int_0^1 \frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{n^3}dx = \sum_{n=1}^{\infty}\frac{1}{n^3}\int_0^1 x^{n-1}\,dx$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^4} = \zeta(4) = \frac{1}{90}\pi^4.$$

and this proves (a).

(b) Notice that, as $x \to 0$, $\frac{\ln(1-x)}{x}$ goes to -1. Therefore when x is very close to 0, the integrand $\frac{\mathrm{Li}_2(x)\ln(1-x)}{x} \approx -\mathrm{Li}_2(x)$. So the integral $\int_0^1 \frac{\mathrm{Li}_2(x)\ln(1-x)}{x}dx$ will behave like the integral of $-\mathrm{Li}_2(x)$ on $[0,1]$, and the integral $\int_0^1 \frac{\mathrm{Li}_2(x)\ln(1-x)}{x}dx$ converges.

The substitution: $\mathrm{Li}_2(x) = u$ immediately leads to the following equality:

$$\int_0^1 \frac{\mathrm{Li}_2(x)\ln(1-x)}{x}dx = -\int_0^{\zeta(2)} u\,du = -\frac{1}{2}\zeta^2(2) = -\frac{1}{72}\pi^4.$$

and this proves (b).

□

Now we are in position to state the theorem.

Theorem 2.3. *The following identity holds:*

$$\sum_{n=1}^{\infty}\frac{1}{n^2}H_n^2 = \frac{17}{360}\pi^4.$$

Proof.

$$\sum_{n=1}^{\infty}\frac{1}{n^2}H_n^2 = \sum_{n=1}^{\infty}\frac{1}{n}H_n^2\int_0^1 x^{n-1}\,dx = \int_0^1\left(\sum_{n=1}^{\infty}\frac{H_n^2}{n}x^{n-1}\right)dx$$
$$= \int_0^1 \frac{\mathrm{Li}_3(x)}{x}dx - \int_0^1 \frac{\mathrm{Li}_2(x)\ln(1-x)}{x}dx - \frac{1}{3}\int_0^1 \frac{\ln^3(1-x)}{x}dx$$
$$= \frac{1}{90}\pi^4 + \frac{1}{72}\pi^4 - \frac{1}{3}\int_0^1 \frac{\ln^3(1-x)}{x}dx = \frac{1}{40}\pi^4 - \frac{1}{3}\int_0^1 \frac{\ln^3(1-x)}{x}dx.$$

Recall ([20, Formula (5), Table 110, p. 159]),

$$\int_0^1 \frac{\ln^{2r-1}(1-x)}{x}dx = \frac{(-1)^r}{r}\,2^{2r-2}\pi^{2r}B_{2r} \quad (r\in\mathbb{N}). \tag{2.2}$$

Setting $r=2$ on both sides of (2.2) yields

$$\int_0^1 \frac{\ln^3(1-x)}{x}dx = 2\,\pi^4 B_4 = -\frac{1}{15}\pi^4. \tag{2.3}$$

With (2.3) in hand, the consequence is:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\,H_n^2 = \frac{1}{40}\pi^4 + \frac{1}{3}\left(\frac{1}{15}\pi^4\right) = \frac{17}{360}\pi^4.$$

which completes the proof. □

Remark 2.4. In the Ref. [20, Formula (5), Table 110, p. 159], the expression for $\int_0^1 \frac{\ln^{2r-1}(1-x)}{x}dx$ has a typo, the correct one being

$$\int_0^1 \frac{\ln^{2r-1}(1-x)}{x}dx = \frac{(-1)^r}{r}\,2^{2r-2}\pi^{2r}B_{2r} \quad (r\in\mathbb{N}).$$

The argument in the second line and fifth line of proofs of Theorem 2.3 and first part of Lemma 2.2 (i.e., interchanging integration and summation) is known as Bernstein's theorem (see [19, Thm. 9.30, p. 243]). It is instructive here to check that (and why) the coefficients are positive.

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METRIZABILITY OF F-METRIC SPACES

SUMIT SOM
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Abstract. Very recently, B. C. Anghelina [Some fixed point theorems in the framework of f-metric spaces, Fixed point theory, 26 (2) (2025), 343-358] introduced the concept of an f-metric space which is a generalization of metric space. In this short note, we show that the newly introduced f-metric space is metrizable.

1. Introduction

In 1906, Fréchet first introduced the notion of a metric space. Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a metric on X if the following conditions are satisfied:

- $d(x, y) = 0$ if and only if $x = y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A topological space (X, τ) is said to be metrizable if there exists a metric $d : X \times X \to [0, \infty)$ on X such that $\tau = \tau_d$ where τ_d denotes the topology induced by the metric d. Authors often generalize the notion of metric spaces by introducing a suitable distance function on a non-empty set. In 2007, Huang and Zhang [4] introduced the notion of a cone metric space by taking the values of a distance function in a real Banach space with respect to a cone in that space. In 2011, Khani and Pourmahdian [5] proved that such cone metric spaces are metrizable. Recently, Anghelina [1] introduced the notion of an f-metric space with respect to a class of functions $f : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$ and $f(x, y) \geq f(x', y')$ for all $x, x', y, y' \in [0, \infty)$ with the property that $x \geq x'$ and $y \geq y'$. Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be an f-metric on X if the following conditions hold:

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- $d(x,y)=0$ if and only if $x=y$;
- $d(x,y)=d(y,x)$ for all $x,y\in X$;
- $d(x,z)\leq f(d(x,y),d(y,z))$ for all $x,y,z\in X$.

If d is an f metric on X then the triplet (X,d,f) is called an f-metric space. In [1], author showed that every metric space (X,d) is an f-metric space for some function $f:[0,\infty)\times[0,\infty)\to[0,\infty)$ satisfying the conditions which we already mentioned above. Firstly, we recall the result from [1].

Theorem 1.1. [1] *Let (X,d) be a metric space and $\theta:[0,\infty)\times[0,\infty)\to[0,\infty)$ such that*

- $\lim\limits_{(x,y)\to(0,0)}\theta(x,y)=0$;
- $\theta(x,y)\geq\theta(x',y')$ *for all* $x,x',y,y'\in[0,\infty)$ *with* $x\geq x'$ *and* $y\geq y'$;
- $x\leq\theta(x,x)$ *for all* $x\in[0,\infty)$;
- $f(x,y)=\theta(x+y,x+y)$ *for all* $x,y\in[0,\infty)$;
- $\rho_d(x,y)=\theta(d(x,y),d(x,y))$.
  Then (X,ρ_d,f) *is an* f*-metric space.*

In this short note, we address the converse of Theorem 1.1. we show that every f-metric space is metrizable. We use metrization theorem due to Niemytski and Wilson in our proof. But still, it is unknown the explicit structure of the metric d with respect to which the space become metrizable. Before proceeding to our metrizability result, we like to recall the metrization theorem due to Niemytski and Wilson (see page 137 in [3]) as follows.

Theorem 1.2. [3] *Let X be a topological space and $F:X\times X\to[0,\infty)$ be a distance function on X. If the distance function F satisfies the following conditions*
(i)$F(x,y)=0$ *if and only if* $x=y$ *for all* $x,y\in X$;
(ii)$F(x,y)=F(y,x)$ *for all* $x,y\in X$;
and any one of the following conditions
(iii-A)given a point $a\in X$ *and a number* $\varepsilon>0$, *there exists* $\phi(a,\varepsilon)>0$ *such that if* $F(a,b)<\phi(a,\varepsilon)$ *and* $F(b,c)<\phi(a,\varepsilon)$ *then* $F(a,c)<\varepsilon$;
(iii-B)if $a\in X$ *and* $\{a_n\}_{n\in\mathbb{N}},\{b_n\}_{n\in\mathbb{N}}$ *are two sequences in* X *such that* $F(a_n,a)\to0$ *and* $F(a_n,b_n)\to0$ *as* $n\to\infty$ *then* $F(b_n,a)\to0$ *as* $n\to\infty$;
(iii-C)for each point $a\in X$ *and positive number* k, *there is a positive number* r *such that if* $b\in X$ *for which* $F(a,b)\geq k$, *and* c *is any point then* $F(a,c)+$

$F(b,c) \geq r$,
then the topological space X is metrizable.

2. Metrizability of f-metric spaces

Now, we show that the newly introduced f-metric space in [1] is metrizable.

Theorem 2.1. *Let (X,d,f) is an f-metric space. Then X is metrizable.*

Proof. Let (X,d,f) is an f-metric space. By the definition of an f-metric space, the distance function d satisfies the first two conditions of the metrization theorem by Niemytski and Wilson. Now, we show that the distance function satisfies condition (iii-B) which is mentioned in Theorem 1.2 (in fact, it can be seen that all the conditions (iii-A),(iii-B),(iii-C) are satified by the f-metric d). Let $a \in X$ and $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}$ be two sequences in X such that $d(a_n,a) \to 0$ and $d(a_n,b_n) \to 0$ as $n \to \infty$. By the third condition of an f-metric space, we have

$$d(b_n,a) \leq f(d(b_n,a_n), d(a_n,a)).$$

So, as $d(a_n,a) \to 0$ and $d(a_n,b_n) \to 0$ as $n \to \infty$, by the property of the function f, we have $f(d(b_n,a_n), d(a_n,a)) \to 0$ as $n \to \infty$. From above, we get $d(b_n,a) \to 0$ as $n \to \infty$. This shows that the f-metric space is metrizable. □

Note 2.2. By applying Chittenden metrization theorem [2], we can also prove the metrizability of an f-metric space. The proof is similar as above and can be done by the property of the function f.

Concluding comments

In [1], author introduced the concept of an f-metric space with respect to a class of functions $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying two conditions. They obtained that if (X,d) is a metric space then (X,ρ_d,f) is an f-metric space for some function f and f-metric ρ_d. In this note, we consider the converse part. We prove that if (X,d,f) is an f-metric space then X is metrizable. But, it is still unknown the form of the explicit metric with respect to which X is metrizable. We left it as an open question.

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DIFFERENTIAL INEQUALITIES FOR POLYNOMIALS WITH RESTRICTED ZEROS

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Abstract. In this paper, we prove some results for a polynomial $P(z)$ of degree n having a multiple zero at some point within the unit disc and the rest of zeros lie either in $|z| \geq k, k \geq 1$ or in $|z| \leq k, k \leq 1$. Our results besides generalizing some Erdös-type inequalities for polynomials also in particular refine a result due to Govil [9] by using simple and elegant technique.

1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials $P(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most n. It was Bernstein [6], who formulated a well-known classical result regarding the estimation of $|P'(z)|$ in terms of $|P(z)|$ on $|z| = 1$ for $z \in \mathbb{C}$ and proved the following:

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The result is sharp and equality holds only when $P(z)$ is a constant multiple of z^n. It is natural to ask what happens to inequality (1.1), if we impose restrictions on the location of zeros of P. In this connection the following inequalities are the earliest belonging to this domain of ideas which have a clear impact on the subsequent work carried forward since then.
If $P(z) \neq 0$ *in* $|z| < 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|, \tag{1.2}$$

whereas in reverse direction, if $P(z) \neq 0$ in $|z| > 1$, then (1.2) can be replaced by

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

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Both the inequalities are sharp and equality in each holds for the polynomials $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. Inequality (1.2) was conjectured by Erdös and latter verified by Lax [11], whereas inequality (1.3) is due to Turán [16].

Malik [12] generalized inequality (1.2) by proving the following:

If $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.4}$$

The above inequality is sharp and extremal polynomial is $P(z) = (z+k)^n$. As a analogous result to (1.4) in case $k \leq 1$, it was proved by Govil [9].

If $P(z) \neq 0$ in $|z| < k$, $k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{1.5}$$

for which $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on the circle $|z| = 1$, where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

2. The Proofs

The following lemmas will be crucial for the proofs of our theorems.

Lemma 2.1. *If $P(z)$ is a polynomial of degree at most n and $Q(z) = z^n\overline{P(1/\bar{z})}$, then for $|z| = 1$*

$$|Q'(z)| + |P'(z)| \leq \frac{n^2}{2}\Big\{M_\alpha + M_{\alpha+\pi}\Big\},$$

where $M_\alpha = \max_{1\leq k\leq n} \big|P\big(e^{\frac{i(\alpha+2k\pi)}{n}}\big)\big|$ and $M_{\alpha+\pi}$ is obtained by replacing α by $\alpha+\pi$.

The proof of Lemma 2.1 is a simple consequence of the principle of mathematical induction.

Lemma 2.2. *If $(y_j)_{j=1}^{\infty}$ be a sequence of real numbers then for all $n \in \mathbb{N}$*

$$\sum_{j=1}^{n} \frac{1-y_j}{1+y_j} \leq \frac{1 - \prod_{j=1}^{n} y_j}{1 + \prod_{j=1}^{n} y_j}, y_j \geq 1. \tag{2.1}$$

The proof of Lemma 2.2 is a simple consequence of the principle of mathematical induction.

Lemma 2.3. *Let* $P(z) = \prod_{j=1}^{n}(z - z_j)$ *be a polynomial of degree* n, *which does not vanish in* $|z| < k, k \le 1$ *and let* $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. *If* $|P'(z)|$ *and* $|Q'(z)|$ *attain maximum at the same point on the circle* $|z| = 1$, *then*

$$\max_{|z|=1}|P'(z)| \le \frac{1}{1+k^n}\left\{n - k^n\sum_{j=1}^{n}\frac{|z_j| - k}{|z_j| + k}\right\}\max_{|z|=1}|P(z)|.$$

The above lemma is due to Aziz et al.[4].
In this paper, we first prove the following result.

Theorem 2.4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n, *which does not vanish in* $|z| < k, k \le 1$ *and let* $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$. *If* $|P'(z)|$ *and* $|Q'(z)|$ *attain maximum at the same point on the circle* $|z| = 1$, *then*

$$\max_{|z|=1}|P'(z)| \le \left\{\frac{n}{1+k^n} - \frac{k^n}{1+k^n}\left(\frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|}\right)\right\}\max_{|z|=1}|P(z)|. \tag{2.2}$$

Remark 2.5. Since $P(z)$ does not vanish in $|z| < k, k \le 1$. Therefore $|a_0| \ge k^n|a_n|$, showing that Theorem 2.4 improves inequality (1.5).

Remark 2.6. For $k = 1$, Theorem 2.4 reduces to a result due Kumar [10].

Proof of Theorem 2.4. By Lemma 2.3, we have

$$|P'(z)| \le \frac{1}{1+k^n}\left(n - k^n\sum_{j=1}^{n}\frac{|z_j| - k}{|z_j| + k}\right)|P(z)|$$
$$= \frac{1}{1+k^n}\left(n - k^n\sum_{j=1}^{n}\frac{\frac{|z_j|}{k} - 1}{\frac{|z_j|}{k} + 1}\right)|P(z)|.$$

Since $\frac{|z_j|}{k} \ge 1$, using Lemma 2.2, we get

$$|P'(z)| \le \frac{1}{1+k^n}\left(n - k^n\prod_{j=1}^{n}\frac{\frac{|z_j|}{k} - 1}{\frac{|z_j|}{k} + 1}\right)|P(z)|$$
$$= \frac{1}{1+k^n}\left(n - k^n\left(\frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|}\right)\right)|P(z)|.$$

That is for $|z| = 1$

$$|P'(z)| \leq \left\{ \frac{n}{1+k^n} - \frac{k^n}{1+k^n} \left(\frac{|a_0| - k^n|a_n|}{|a_0| + k^n|a_n|} \right) \right\} |P(z)|.$$

This completes the proof of Theorem 2.4. □

We next prove the following generalization of Theorem 2.4.

Theorem 2.7. *Let* $P(z) = z^s \left(\sum_{j=0}^{n-s} a_j z^j \right)$ *be a polynomial of degree* n, *having all zeros in* $|z| \geq k, k \leq 1$ *except a zero at the origin of multiplicity* $s \geq 0$. *Also let both* $|P'(z)|$ *and* $|Q'(z)|$ *attain maximum at the same point on the circle* $|z| = 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \left\{ \frac{n + sk^{n-s}}{1+k^{n-s}} - \frac{k^{n-s}}{1+k^{n-s}} \left(\frac{|a_0| - k^{n-s}|a_{n-s}|}{|a_0| + k^{n-s}|a_{n-s}|} \right) \right\} \max_{|z|=1} |P(z)|. \tag{2.3}$$

For $s = 0$, Theorem 2.7 reduces to Theorem 2.4.

Proof of Theorem 2.7. Let $P(z) = z^s \phi(z)$, where $\phi(z) = a_0 + \sum_{j=1}^{n-s} a_j z^j$ is a polynomial of degree $n - s$, which does not vanish in $|z| < k, k \leq 1$, then using (2.2), we get

$$|\phi'(z)| \leq \left\{ \frac{n-s}{1+k^{n-s}} - \frac{k^{n-s}}{1+k^{n-s}} \left(\frac{|a_0| - k^{n-s}|a_{n-s}|}{|a_0| + k^{n-s}|a_{n-s}|} \right) \right\} |\phi(z)|. \tag{2.4}$$

Now we have

$$zP'(z) = sz^s\phi(z) + z^{s+1}\phi'(z) = sP(z) + z^{s+1}\phi'(z).$$

For $|z| = 1$, we have

$$|P'(z)| \leq s|P(z)| + |\phi'(z)|. \tag{2.5}$$

Combining (2.4) and (2.5), we get for $|z| = 1$

$$|P'(z)| \leq s|P(z)| + \left\{ \frac{n-s}{1+k^{n-s}} - \frac{k^{n-s}}{1+k^{n-s}} \left(\frac{|a_0| - k^{n-s}|a_{n-s}|}{|a_0| + k^{n-s}|a_{n-s}|} \right) \right\} |\phi(z)|.$$

Now using $\max_{|z|=1} |P(z)| = \max_{|z|=1} |\phi(z)|$, we get

$$\max_{|z|=1} |P'(z)| \leq \left\{ \frac{n + sk^{n-s}}{1+k^{n-s}} - \frac{k^{n-s}}{1+k^{n-s}} \left(\frac{|a_0| - k^{n-s}|a_{n-s}|}{|a_0| + k^{n-s}|a_{n-s}|} \right) \right\} \max_{|z|=1} |P(z)|.$$

This completes the proof of Theorem 2.7. □

Further for the polynomials $P \in \mathscr{P}_{n,\mu}$, where $\mathscr{P}_{n,\mu} := \left\{P : P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \le \mu \le n\right\}$ instead of assuming a zero of order s at the origin, we assume that there is a zero at a point z_0 of order s with $|z_0| < 1$ and prove the following generalizations of (1.2) and (1.4).

Theorem 2.8. *If $P \in \mathscr{P}_{n,\mu}$, such that $P(z) = (z - z_0)^s \left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\right)$, $1 \le \mu \le n - s$, $0 \le s \le n - 1$, has no zero in $|z| \le k$, $k > \mu$, $\mu \ge 1$, except a zero of multiplicity s, $0 \le s < n$ at z_0, where $|z_0| \le 1 - \eta$ and $n > n_0$, then for $|z| = 1$*

$$|P^{'}(z)| \le \frac{1}{2}\left[n^2 - \frac{1}{M_\alpha^2 + M_{\alpha+\pi}^2}\left\{n^2 - 2n\left(\frac{s}{1-|z_0|} + \frac{\mu(n-s)}{\mu+k}\right)|P(z)|^2\right\}\right]^{\frac{1}{2}}$$
$$\left\{M_\alpha^2 + M_{\alpha+\pi}^2\right\}^{\frac{1}{2}}.$$

where $\eta = \dfrac{2s(k+\mu)}{n(k-\mu)+2s\mu}$, $n_0 = \dfrac{2ks}{k-\mu}$ and M_α, $M_{\alpha+\pi}$ are defined in Lemma 2.1.

Taking $z_0 = 0$, Theorem 2.8 reduces to the following.

Corollary 2.9. *If $P \in \mathscr{P}_{n,\mu}$, such that $P(z) = z^s \left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\right)$, $1 \le \mu \le n - s$, $0 \le s \le n - 1$, has no zero in $|z| \le k$, $k > \mu$, $\mu \ge 1$, except a zero of multiplicity s, $0 \le s < n$ at $z = 0$, then for $n > n_0$ and $|z| = 1$*

$$|P^{'}(z)| \le \frac{1}{2}\left[n^2 - \frac{1}{M_\alpha^2 + M_{\alpha+\pi}^2}\left\{n^2 - 2n\left(\frac{ks+n\mu}{\mu+k}\right)|P(z)|^2\right\}\right]^{\frac{1}{2}}$$
$$\left\{M_\alpha^2 + M_{\alpha+\pi}^2\right\}^{\frac{1}{2}}.$$

Remark 2.10. Since for $|z_0| \le 1 - \eta$ and $n > n_0$,

$$\left\{n^2 - 2n\left(\frac{s}{1-|z_0|} + \frac{\mu(n-s)}{\mu+k}\right)|P(z)|^2 \ge 0.\right\}.$$

Therefore Theorem 2.8 improves a result due to Aziz [3].

Remark 2.11. If we take $\mu = 1$ in Theorem 2.8, we get an improvement of a result due to Ahanger et al. [1].

Proof of Theorem 2.8. Since $P(z)$ has all its zeros in $|z| \geq k$ and a zero of multiplicity s at z_0, $|z_0| \leq 1-\epsilon < 1$, $0 \leq s < n$, we have

$$P(z) = (z - z_0)^s u(z), \tag{2.6}$$

where $u(z) = \left(a_0 + \sum_{j=\mu}^{n-s} a_j z^j\right)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, is a polynomial of degree at most $n-s$ having all zeros in $|z| \geq k$. Therefore, if $z_1, z_2, \ldots, z_{n-s}$ be the zeros of $u(z)$, then $|z_j| \geq k,\ k > 1$, $j \in \{1, 2, \ldots, n-s\}$, we have from (2.6)

$$\frac{zP'(z)}{P(z)} = \frac{sz}{z - z_0} + \sum_{j=1}^{n-s} \frac{z}{z - z_j}.$$

This, in particular, gives

$$\operatorname{Re}\left(\frac{zP'(z)}{P(z)}\right) = \operatorname{Re}\left(\frac{sz}{z - z_0}\right) + \operatorname{Re}\left(\sum_{j=1}^{n-s} \frac{z}{z - z_j}\right).$$

For the points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are not the zeros of $P(z)$, we have

$$\begin{aligned}\operatorname{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) &= \operatorname{Re}\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) + \operatorname{Re}\left(\sum_{j=1}^{n-s} \frac{e^{i\theta}}{e^{i\theta} - z_j}\right) \\ &= \operatorname{Re}\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) + \operatorname{Re}\left(\sum_{j=1}^{n-s} \frac{1}{1 - e^{-i\theta} z_j}\right). \end{aligned} \tag{2.7}$$

Using the fact that for $|w| \geq k$, $k > 1$, and $\mu \geq 1$,

$$\operatorname{Re}\left(\frac{1}{1-w}\right) \leq \frac{1}{1+k} \leq \frac{\mu}{\mu + k}$$

and

$$\operatorname{Re}\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) \leq \left|\frac{se^{i\theta}}{e^{i\theta} - z_0}\right| \leq \frac{s}{1 - |z_0|}.$$

We get from (2.7), for $0 \leq \theta < 2\pi$,

$$\operatorname{Re}\left(\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right) \leq \frac{s}{1 - |z_0|} + \frac{\mu(n-s)}{\mu + k}.$$

Now, for $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$ and $|z|=1$, it can be easily verified that

$$|Q'(z)|=|nP(z)-zP'(z)|.$$

This gives, for $|z|=1$,

$$\begin{aligned}\left|\frac{Q'(z)}{P(z)}\right|^2 &= \left|n-\frac{zP'(z)}{P(z)}\right|^2\\ &= n^2+\left|\frac{zP'(z)}{P(z)}\right|^2-2n\mathrm{Re}\left(\frac{\mathrm{zP'(z)}}{\mathrm{P(z)}}\right)\\ &\geq n^2+\left|\frac{zP'(z)}{P(z)}\right|^2-2n\left(\frac{s}{1-|z_0|}+\frac{\mu(n-s)}{\mu+k}\right).\end{aligned}$$

That is

$$|Q'(z)|^2\geq|zP'(z)|^2+\left\{n^2-2n\left(\frac{s}{1-|z_0|}+\frac{\mu(n-s)}{\mu+k}\right)\right\}|P(z)|^2. \qquad (2.8)$$

For $|z_0|\leq 1-\eta$ and $n\geq n_0$, where $\eta=\dfrac{2s(k+\mu)}{n(k-\mu)+2\mu s}$ and $n_0=\dfrac{2sk}{(k-\mu)}$ with $k>\mu$, $\mu\geq 1$, it can be easily verified that

$$n^2-2n\left(\frac{s}{1-|z_0|}+\frac{\mu(n-s)}{\mu+k}\right)\geq 0.$$

Therefore from (2.8), we have for $|z_0|\leq 1-\eta$, $n\geq n_0$ and $|z|=1$

$$|Q^{'}(z)|^2\geq|P^{'}(z)|^2+\left\{n^2-2n\left(\frac{s}{1-|z_0|}+\frac{\mu(n-s)}{\mu+k}\right)\right\}|P(z)|^2. \qquad (2.9)$$

Now using inequality (2.9) in Lemma 2.1, we get

$$|P^{'}(z)|^2+\left\{n^2-2n\left(\frac{s}{1-|z_0|}+\frac{\mu(n-s)}{\mu+k}\right)\right\}|P(z)|^2\leq\frac{n^2}{4}\left\{M_\alpha^2+M_{\alpha+\pi}^2\right\}.$$

Equivalently, we get

$$\begin{aligned}|P^{'}(z)|^2\leq&\frac{n^2}{4}\left\{M_\alpha^2+M_{\alpha+\pi}^2\right\}\\ &-\frac{1}{2}\left\{n^2-2n\left(\frac{s}{1-|z_0|}+\frac{\mu(n-s)}{\mu+k}\right)|P(z)|^2\right\}.\end{aligned}$$

This gives for $|z| = 1$

$$|P'(z)| \leq \frac{1}{2}\left[n^2 - \frac{1}{M_\alpha^2 + M_{\alpha+\pi}^2}\left\{n^2 - 2n\left(\frac{s}{1-|z_0|} + \frac{\mu(n-s)}{\mu+k}\right)|P(z)|^2\right\}\right]^{\frac{1}{2}}$$

$$\left\{M_\alpha^2 + M_{\alpha+\pi}^2\right\}^{\frac{1}{2}}.$$

This completes the proof of Theorem 2.8 □

For a polynomial $P(z)$ of degree n, the polar derivative of $P(z)$ denoted by $D_\alpha P(z)$, is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

It is to be observed that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Aziz [2] obtained the following result to the polar derivative of a polynomial by proving the following:

If $P(z)$ is a polynomial of degree n, such that $P(z)$ does not vanish in $|z| < k, k \geq 1$, then for every real or complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leq n\left(\frac{k+|\alpha|}{1+k}\right) \max_{|z|=1} |P(z)|. \tag{2.10}$$

These inequalities have been extended and generalized from time to time by various authors (for reference see [14]). The extension of inequalities from ordinary derivative to polar derivative of a polynomial is one of the extension and widely studied topic and over the years several refinements and generalizations of above inequalities have been proved by introducing restrictions on the multiplicity of zero at $z = 0$. So it is natural to ask what happens in case the polynomial $P(z)$ of degree n having a zero of some multiplicity at origin and the rest of the zeros in $|z| \geq k, k \leq 1$. As an answer to this question, here we prove the following results which in particular extend some inequalities to such a class of polynomials.

Theorem 2.12. *Let $P(z) = z^s\left(\sum_{j=0}^{n-s} a_j z^j\right)$ be a polynomial of degree n, having all zeros in $|z| \geq k, k \leq 1$, except a zero of multiplicity $s \geq 0$ at the origin. Let $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on the* *circle* $|z| = 1$, *and for any* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geq 1$,

$$|D_\alpha P(z)| \leq n \max_{|z|=1} |P(z)| + (|\alpha| - 1) \left\{ \frac{n + sk^{n-s}}{1 + k^{n-s}} - \frac{k^{n-s}}{1 + k^{n-s}} \left(\frac{|a_0| - k^{n-s}|a_{n-s}|}{|a_0| + k^{n-s}|a_{n-s}|} \right) \right\} \max_{|z|=1} |P(z)|. \tag{2.11}$$

If we divide both sides of inequality (2.11) by $|\alpha|$, it reduces to an inequality (2.3).

Remark 2.13. By taking $s = 0$ and since $|a_0| \geq k^n|a_n|$, showing that Theorem 2.12 improves a result due to Chanam [7].

Proof of Theorem 2.12. Since for $|z| = 1$, $|Q'(z)| = |nP(z) - zP'(z)|$. Therefore for $|z| = 1$

$$\begin{aligned} |D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &= |nP(z) - zP'(z) + \alpha P'(z)| \\ &\leq |nP(z) - zP'(z)| + |\alpha||P'(z)| \\ &= |Q'(z)| + |\alpha||P'(z)| \\ &= |P'(z)| + |Q'(z)| + (|\alpha| - 1)|P'(z)| \\ &\leq n \max_{|z|=1} |P(z)| + (|\alpha| - 1) \max_{|z|=1} |P'(z)|. \end{aligned}$$

Now on using inequality 2.3, we get

$$|D_\alpha P(z)| \leq n \max_{|z|=1} |P(z)| + (|\alpha| - 1) \left\{ \frac{n + sk^{n-s}}{1 + k^{n-s}} - \frac{k^{n-s}}{1 + k^{n-s}} \left(\frac{|a_0| - k^{n-s}|a_{n-s}|}{|a_0| + k^{n-s}|a_{n-s}|} \right) \right\} \max_{|z|=1} |P(z)|.$$

This completes the proof of Theorem 2.12. □

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ON GLAISHER'S CONGRUENCES

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Abstract. This paper offers derivations for two congruences attributed to J.W.L. Glaisher, one of which is a generalization of Wolstenholme's congruence. To maintain a self-contained exposition, we include comprehensive proofs of Faulhaber's formula, Newton's formula, and the von Staudt-Clausen theorem, each playing an indispensable role in our proofs of Glaisher's congruences.

1. The motivation behind this article

This article's journey began with a curious observation. The second author, while crafting questions on Wolstenholme's congruence for a number theory course, noticed that the congruence (2.10) held true not just for the specific case of

$$2r + 1 = p - 2,$$

but for a broader range, namely,

$$2 \leq r < (p-1)/2.$$

However, the standard methods for proving Wolstenholme's congruence did not seem to apply to this expanded scenario. This puzzle led him to consult the first author, and their collaborative search for a proof eventually guided them to the foundational works of J.W.L. Glaisher and L. Carlitz. The goal of this article is to provide a self-contained exploration, laying out the derivations of several interesting congruences, including (2.8), (2.9), (7.5), and (7.6).

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2. Introduction

The well-known Wilson's theorem states that for any odd prime p,

$$(p-1)! \equiv -1 \pmod p. \tag{2.1}$$

There are many proofs of (2.1) in the literature. One of the proofs involves Fermat's Little Theorem, which can be stated as

$$[j]_p^{p-1} = [1]_p$$

for $1 \leq j \leq p-1$, where

$$[j]_p = \{m \in \mathbf{Z} | m \equiv j \pmod p\}.$$

This implies that for $1 \leq j \leq p-1$, $[j]_p$ are the zeroes of $x^{p-1} - [1]_p$. Therefore,

$$x^{p-1} - [1]_p = (x-[1]_p)(x-[2]_p)\cdots(x-[p-1]_p) \tag{2.2}$$

in $\mathbf{F}_p[x]$, where $\mathbf{F}_p$ is the finite field with p elements. By comparing the constant terms of both sides of the above identity, we deduce (2.1) immediately.

Let

$$\begin{aligned}(x-x_1)(x-x_2)\cdots(x-x_n) &= x^n - A_{n,1}(x_1,x_2,\cdots,x_n)x^{n-1} + \cdots \\ &+ (-1)^{n-1}A_{n,n-1}(x_1,x_2,\cdots,x_n)x + (-1)^n A_{n,n}\end{aligned} \tag{2.3}$$

where

$$A_{n,j}(x_1,x_2,\cdots,x_n) = \sum_{1\leq i_1<i_2<\cdots<i_j\leq n} x_{i_1}\cdots x_{i_j},$$

with $1 \leq j \leq n$. When $x_j = j$ and $n = p-1$, we write

$$\begin{aligned}&(x-1)(x-2)\cdots(x-(p-1)) \\ &= x^{p-1} - a_{p-1,1}x^{p-2} + \cdots + (-1)^{p-2}a_{p-1,p-2}x + (-1)^{p-1}a_{p-1,p-1},\end{aligned} \tag{2.4}$$

where

$$a_{p-1,j} = A_{p-1,j}(1,2,\cdots,p-1) = \sum_{1\leq i_1<i_2<\cdots<i_j\leq n} i_1\cdots i_j,$$

with $1 \leq j \leq p-1$.

Let

$$\overline{a}_{p-1,j} = A_{p-1,j}([1]_p,[2]_p,\cdots,[p-1]_p).$$

Using (2.2) and (2.4), we conclude that

$$x^{p-1} - [1]_p =$$
$$x^{p-1} - \overline{a}_{p-1,1}x^{p-2} + \cdots + (-1)^{p-2}\overline{a}_{p-1,p-2}x + (-1)^{p-1}\overline{a}_{p-1,p-1}, \quad (2.5)$$

which immediately implies that for $1 \le \ell < p-1$,

$$\overline{a}_{p-1,\ell} = [0]_p$$

in $\mathbf{F}_p$. This is equivalent to

$$a_{p-1,\ell} \equiv 0 \pmod{p}. \quad (2.6)$$

Let p be a prime greater than 3. Let $x = p$ in (2.4) to deduce that

$$(p-1)! = p^{p-1} - a_{p-1,1}p^{p-2} + \cdots + a_{p-1,p-3}p^2 - a_{p-1,p-2}p + (p-1)!.$$

Subtracting $(p-1)!$ from both sides, followed by dividing by p, we deduce using (2.6), the Wolstenholme congruence

$$a_{p-1,p-2} \equiv 0 \pmod{p^2}. \quad (2.7)$$

For integer $n \ge 1$, the Bernoulli number B_n is defined as the n-th derivative of the function

$$\frac{x}{e^x - 1}$$

evaluated at $x = 0$. In other words,

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!}x^j.$$

Around 1900, Glaisher [4] showed that for $1 \le r \le (p-1)/2$,

$$a_{p-1,2r} \equiv -\frac{p}{2r}B_{2r} \pmod{p^2} \quad (2.8)$$

and

$$a_{p-1,2r+1} \equiv p^2\frac{2r+1}{4r}B_{2r} \pmod{p^3}. \quad (2.9)$$

It is known from the von Staudt-Clausen congruence that the denominator of B_{2r} is divisible by primes q for which $(q-1)|2r$. Since $p > 2r$, (2.9) implies that

$$a_{p-1,2r+1} \equiv 0 \pmod{p^2}. \quad (2.10)$$

Note that (2.7) is a special case of (2.10) and for $p \ge 5$, $a_{p-1,p-2}$ is not the only number that vanishes modulo p^2.

In the following sections, using Newton's formula, Faulhaber's identity and the von Staudt-Clausen congruence, we provide a proof of (2.10), followed by proofs of (2.8) and (2.9).

3. The Faulhaber identity for the sum of consecutive k-th power

Let $B_k(t)$ be defined as the n-th derivative of the function

$$\frac{xe^{tx}}{e^x-1}.$$

In other words,

$$\frac{xe^{tx}}{e^x-1}=\sum_{j=0}^{\infty}B_j(t)\frac{x^j}{j!}. \tag{3.1}$$

Observe that

$$\frac{xe^{tx}}{e^x-1}=\frac{x}{e^x-1}e^{tx}=\left(\sum_{j=0}^{\infty}\frac{B_j}{j!}x^j\right)\left(\sum_{k=0}^{\infty}t^k\frac{x^k}{k!}\right).$$

Therefore,

$$B_k(t)=\sum_{j=0}^{k}\binom{k}{j}t^jB_{k-j}.$$

Next, observe that

$$xe^{tx}=xe^{tx}\frac{e^x-1}{e^x-1}=\frac{xe^{(t+1)x}}{e^x-1}-\frac{xe^{tx}}{e^x-1}.$$

By using (3.1), we conclude that

$$t^k=\frac{B_{k+1}(t+1)-B_{k+1}(t)}{k+1}.$$

Therefore,

$$\begin{aligned}\sum_{t=1}^{n}t^k&=\frac{B_{k+1}(n+1)-B_{k+1}(0)}{k+1}\\&=\frac{1}{k+1}\left(\sum_{j=0}^{k}\binom{k+1}{j}(n+1)^{k+1-j}B_j\right).\end{aligned} \tag{3.2}$$

This is known as the Faulhaber identity [2, p. 106].

Let $n=p-1$ and

$$s_k=\sum_{t=1}^{p-1}t^k. \tag{3.3}$$

From (3.2), we find that

$$s_k = \frac{1}{k+1}\left(\sum_{j=0}^{k}\binom{k+1}{j}p^{k+1-j}B_j\right). \tag{3.4}$$

Since $B_{2r+1}=0$ for $r\geq 1$, we conclude that for $r\geq 2$,

$$s_{2r}\equiv pB_{2r} \pmod{p^3} \tag{3.5}$$

and

$$s_{2r+1}\equiv \frac{2r+1}{2}p^2B_{2r} \pmod{p^4}. \tag{3.6}$$

Congruences (3.5) and (3.6) are due to Carlitz [1, (2.4)].

Next, by the von Staudt-Clausen Theorem [6, p. 49], we know that

$$B_{2r}+\sum_{(q-1)|2r}\frac{1}{q}\in \mathbf{Z},$$

where each q is a prime. The above property of B_{2r} implies that if B_{2r} is written as a proper fraction n_{2r}/d_{2r} and $q|d_{2r}$ then $(q-1)\leq 2r$. Therefore, if $p-1>2r$ then the d_{2r} is not divisible by p. Hence, for $p-1>2r$,

$$s_{2r}\equiv 0 \pmod{p} \tag{3.7}$$

and

$$s_{2r+1}\equiv 0 \pmod{p^2}. \tag{3.8}$$

For the completeness of this article, we will present a proof of the von Staudt-Clausen Theorem in Section 6.

4. The Newton formula for the sum of consecutive k-th power

For this section, we follow the presentation of [3, Chapter 2]. A polynomial $f(x_1,x_2,\cdots,x_n)$ is symmetric if

$$f(x_{\sigma(1)},x_{\sigma(2)},\cdots,x_{\sigma(n)})=f(x_1,x_2,\cdots,x_n)$$

for all $\sigma\in\mathcal{S}_n$ where $\mathcal{S}_n$ is the symmetric group of degree n. It is clear that if f and g are symmetric, then $af+bg$ is symmetric for any $a,b\in\mathbf{C}$.

Given any positive integer N, there are only finitely many positive integers $k_1,k_2,\cdots,k_n$, such that $k_1+k_2+\cdots+k_n=N$. Note that $k_j\leq N$ and so there are at most N choices for each k_j. Therefore, the number of terms of the form $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$ with $k_1+k_2+\cdots+k_n=N$ is finite. We now order the monomials $x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$ as follow: We say that

$$x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n} < x_1^{\ell_1}x_2^{\ell_2}\cdots x_n^{\ell_n}$$

if $k_1+\cdots+k_n < \ell_1+\cdots+\ell_n$

or

if $k_1+\cdots+k_n = \ell_1+\cdots+\ell_n$ and $k_1 < \ell_1$,

or

if $k_1+\cdots+k_n = \ell_1+\cdots+\ell_n$ and $k_1 = \ell_1, k_2 < \ell_2$,

or

$\vdots$

or

if $k_1+\cdots+k_n = \ell_1+\cdots+\ell_n$ and $k_1 = \ell_1, \cdots, k_{n-2} = \ell_{n-2}, k_{n-1} < \ell_{n-1}$.

For example, $x_1^4x_2^2x_3 < x_1^2x_2^3x_3^3$ and $x_1^4x_2x_3^2 < x_1^4x_2^2x_3$.

With this ordering on monomials, each polynomial $f \in F[x_1,\cdots,x_n]$ has a monomial that is "greater than" the rest of the monomials in f. We call this monomial the leading term of f and denote it as $LT(f)$. For example,

$$LT(x_1x_2+x_2x_3+x_1x_3)$$

is x_1x_2 and

$$LT(A_{n,j}) = x_1x_2\cdots x_j$$

where $A_{n,j}, 1\le j\le n-1$, is given by (2.3).

Let

$$S_r = S_r(x_1,x_2,\cdots,x_n) = \sum_{j=1}^{n} x_j^r,$$

with $1\le r\le n$. Note that s_k defined in (3.3) is $s_r = S_r(1,2,\cdots,n)$. It turns out that for $r\ge 2$,

$$S_r = A_{n,1}S_{r-1} - A_{n,2}S_{r-2} + \cdots + (-1)^{r-2}A_{n,r-1}S_1 + (-1)^{r-1}rA_{n,r}. \quad (4.1)$$

To prove (4.1), we observe that

$$LT(S_r - A_{n,1}S_{r-1}) = -x_1^{r-1}x_2.$$

This leading term can be removed using $A_{n-1,2}S_{r-2}$, giving

$$LT(S_r - A_{n,1}S_{r-1} + A_{n-1,2}S_{r-2}) = x_1^{r-2}x_2x_3.$$

Note that we may continue with this process and the power of x_1 in the leading term of the resulting polynomial reduces by 1 in each step. Continuing this process, we will eventually arrive at

$$LT(S_r - A_{n,1}S_{r-1} + A_{n,2}S_{r-2} + \cdots - (-1)^{r-2}A_{n,r-1}S_1) = c(-1)^{r-1}x_1x_2\cdots x_n$$

for some constant c. Using the fact that sums and differences of symmetric polynomials are symmetric polynomials, we deduce that that

$$S_r = A_{n,1}S_{r-1} - A_{n,2}S_{r-2} + \cdots + (-1)^{r-1}cA_{n,r}. \tag{4.2}$$

To complete the proof of (4.1), it remains to determine c. If we set $x_j = 1$ for all $1 \le j \le r$, then from (4.2), we find that

$$\sum_{j=0}^{r-1}(-1)^j \binom{n}{j} = (-1)^{r-1}\frac{c}{n}\binom{n}{r}. \tag{4.3}$$

On the other hand, by comparing the coefficient of z^{r-1} in the expansion of the left-hand side and the right-hand side of the identity

$$\frac{(1+z)^n}{1+z} = (1+z)^{n-1},$$

we deduce that

$$\sum_{j=0}^{r-1}(-1)^{r-j-1}\binom{n}{j} = \binom{n-1}{r-1} = \frac{r}{n}\binom{n}{r}. \tag{4.4}$$

Using (4.3) and (4.4), we deduce that $c = r$ and the proof of (4.1) is complete.

5. Proofs of (2.8)–(2.10)

First, recall from (2.6) that for $1 \le \ell \le p-1$,

$$a_{p-1,\ell} \equiv 0 \pmod{p}.$$

Replacing r by $2r+1$ and n by $p-1$ in (4.1), we obtain

$$(2r+1)a_{p-1,2r+1} = s_{2r+1} - a_{p-1,1}s_{2r} + a_{p-1,2}s_{2r-1} + \cdots + a_{p-1,2r}s_1. \tag{5.1}$$

Next, by (3.8) and (3.7), we have, for $1 \le j < p-1$,

$$s_j \equiv 0 \pmod{p}.$$

Therefore, we deduce from (5.1) that for $r \geq 1$,

$$(2r+1)a_{p-1,2r+1} \equiv s_{2r+1} \pmod{p^2},$$

which implies that

$$a_{p-1,2r+1} \equiv 0 \pmod{p^2}$$

by (3.8).

Our next task is to prove (2.8) and (2.9). We proceed by verifying that (2.8) holds for $r = 1$. We then show that if (2.8) is true then (2.9) is true. This is followed by showing that (2.9) implies

$$a_{p-1,2r+2} \equiv -\frac{p}{2r+2} B_{2r+2} \pmod{p^2}. \tag{5.2}$$

Completing the above steps will imply that (2.8) and (2.9) hold for all $r \geq 1$.

By setting $r = 2$ and $n = p - 1$ in (4.1), we obtain

$$2a_{p-1,2} = s_2 - a_{p-1,1}s_1 = \frac{(p-1)p(2p-1)}{6} - \frac{p^2(p-1)^2}{4},$$

which immediately implies that

$$a_{p-1,2} \equiv -\frac{p}{2} B_2 \pmod{p^2},$$

which is (2.8) for $r = 1$. Next, from (3.4), we deduce that for $r \geq 1$,

$$s_{2r} = pB_{2r} + p^2 K_{2r} \tag{5.3}$$

and

$$s_{2r+1} = \frac{2r+1}{2} p^2 B_{2r} + p^3 K_{2r+1} \tag{5.4}$$

for some integers K_{2r} and K_{2r+1}. Note that for $r \geq 2$, (5.3) and (5.4) are weaker than (3.5) and (3.6).

Suppose (2.8) holds for all positive even integers less than or equal to $2r$ and (2.9) holds for all odd integers greater than 1 but less than $2r - 1$. Rewrite (2.8) as

$$a_{p-1,2r} = -\frac{pB_{2r}}{2r} + p^2 L_{2r} \tag{5.5}$$

for some integer L_{2r}. From (5.1), we have

$$(2r+1)a_{p-1,2r+1} = s_{2r+1} - a_{p-1,1}s_{2r} + \cdots + (-1)^{2r} a_{p-1,2r}s_1.$$

The terms $a_{p-1,\nu}s_{2r+1-\nu}$, for $2 \le \nu \le 2r-1$, are all divisible by p^3 by applying (2.8) and (2.9) with $\nu \le 2r$ and using both (3.5) and (3.6). Therefore,

$$\begin{aligned}(2r+1)a_{p-1,2r+1} &\equiv \frac{2r+1}{2}p^2B_{2r} + p^3K_{2r+1} - \frac{p(p-1)}{2}\left(pB_{2r}+p^2K_{2r}\right) \\ &\quad + \left(-\frac{p}{2r}B_{2r}+p^2L_{2r}\right)\frac{p(p-1)}{2} \pmod{p^3} \\ &\equiv B_{2r}\left(\frac{2r+1}{2}p^2+\frac{p^2}{2}+\frac{p^2}{4r}\right) \pmod{p^3}.\end{aligned}$$

Simplifying the above yields (2.9). To show that (2.9) implies (2.8) with $2r$ replaced by $2r+2$, we use (4.1) to write

$$\begin{aligned}(2r+2)a_{p-1,2r+2} &= -s_{2r+2} + a_{p-1,1}s_{2r+1} - \cdots + a_{p-1,2r+1}s_1 \\ &\equiv -pB_{2r+2} \pmod{p^2}\end{aligned}$$

by (5.4). Therefore, (2.8) and (2.9) holds for all integers $r \ge 1$.

6. The von Staudt-Clausen Theorem

In this section, we present a proof of the von Staudt-Clausen Theorem. This proof shows the close connection between the von Staudt-Clausen Theorem and the converse of Wilson's Theorem. Our proof is a modification of the proof given in [6, p. 49]. We hope to give a clearer presentation of the proof of the von Staudt-Clausen Theorem using (6.5).

Wilson's Theorem states that if $p > 2$ is a prime, then

$$(p-1)! \equiv -1 \pmod{p}.$$

It turns out that the converse of Wilson's Theorem is true. In other words, if n is composite then

$$(n-1)! \not\equiv -1 \pmod{n}.$$

To prove this, we first verify it for $n = 4$. In this case $3! \equiv 6 \equiv 2 \pmod 4$ and the statement is true. Suppose $n \ge 6$ is composite. Then $n = ab$. If $2 \le a < b$ then a and b are both less than $ab-1$ and therefore $ab|(ab-1)!$. Suppose $a = b$ with $a \ge 3$. Then $ab - 1 = a^2 - 1 > a^2 - a = a(a-1) \ge 2a$. So

$$(a^2-1)! \equiv 0 \pmod{a^2}.$$

We therefore conclude that if $n \ge 6$ is composite then

$$(n-1)! \equiv 0 \pmod{n}.$$

Now, from the definition of B_ℓ, we conclude that

$$B_\ell = f^{(\ell)}(0),$$

where $f(x) = x/(e^x - 1)$. Write

$$f(x) = \frac{\ln(1-(1-e^x))}{e^x - 1} = \sum_{j=1}^{\infty} \frac{(1-e^x)^{j-1}}{j}.$$

Note that

$$B_\ell = \sum_{j=1}^{\infty} \frac{C_{j,\ell}}{j}, \tag{6.1}$$

where

$$C_{j,\ell} = g_j^{(\ell)}(0)$$

with

$$g_j(x) = (1-e^x)^{j-1}.$$

First, note that $C_{1,\ell} = 1$. Next, for $j > 1$,

$$g_j^{(1)}(x) = -e^x(j-1)(1-e^x)^{j-2}. \tag{6.2}$$

For $j = 2$, we find that $C_{2,\ell} = -1$ and therefore, by (6.1), we deduce that

$$B_\ell = 1 - \frac{1}{2} + \sum_{j=3}^{\infty} \frac{C_{j,\ell}}{j}. \tag{6.3}$$

Next, from (6.2), we deduce that for $j > 2$,

$$g_j^{(2)}(x) = -e^x(j-1)(1-e^x)^{j-2} + (j-1)(j-2)(-e^x)^2(1-e^x)^{j-3},$$

and this can be written as

$$g_j^{(2)}(x) - g_j^{(1)}(x) = (j-1)(j-2)(-e^x)^2(1-e^x)^{j-3} \tag{6.4}$$

using (6.2).

We next show by induction on s that for $2 \le s \le j-1$,

$$\begin{aligned} g_j^{(s)}(x) &+ c_{j,s-1}g_j^{(s-1)}(x) + \cdots + c_{j,1}g_j^{(1)}(x) \\ &= (j-1)(j-2)\cdots(j-s)(-e^x)^s(1-e^x)^{j-(s+1)}, \end{aligned} \tag{6.5}$$

where $c_{j,\nu}$ with $1 \le \nu \le s-1$ are constants independent of x. The case $s = 2$ is precisely (6.4). Suppose (6.5) holds for some integer s. Then differentiating both sides of (6.5) with respect to x, we find that

$$\begin{aligned} g_j^{(s+1)}(x) &+ c_{j,s-1} g_j^{(s)}(x) + \cdots + c_{j,1} g_j^{(2)}(x) \\ &= s(j-1)(j-2)\cdots(j-s)(-e^x)^s(1-e^x)^{j-(s+1)} \\ &\quad + (j-1)(j-2)\cdots(j-s)(j-s-1)(-e^x)^{s+1}(1-e^x)^{j-(s+2)}. \end{aligned} \tag{6.6}$$

By (6.5), the first term on the right-hand side of (6.6) can be written as

$$\begin{aligned} s(j-1)&(j-2)\cdots(j-s)(-e^x)^s(1-e^x)^{j-(s+1)} \\ &= s\left(g_j^{(s)}(x) + c_{j,s-1} g_j^{(s-1)}(x) + \cdots + c_{j,1} g_j^{(1)}(x)\right). \end{aligned}$$

Hence, we may rewrite (6.6) as

$$\begin{aligned} g_j^{(s+1)}(x) &+ (c_{j,s-1} - s) g_j^{(s)}(x) + (c_{j,s-2} - s c_{j,s-1}) g_j^{(s-1)}(x) + \\ &\cdots + (c_{j,1} - s c_{j,2}) g_j^{(2)}(x) - s c_{j,1} g_j^{(1)}(x) \\ &= (j-1)(j-2)\cdots(j-s)(j-s-1)(-e^x)^{s+1}(1-e^x)^{j-(s+2)}, \end{aligned}$$

and (6.5) is true for $s+1$.

Observe from (6.5) that when $x = 0$ and $1 \le s < j-1$,

$$g_j^{(s)}(0) = 0.$$

Therefore, for $j > \ell + 1$, $C_{j,\ell} = g_j^{(\ell)}(0) = 0$ and the right-hand side of (6.1) is a finite sum and

$$B_\ell = 1 - \frac{1}{2} + \sum_{j=3}^{\ell+1} \frac{C_{j,\ell}}{j}. \tag{6.7}$$

Next, for $s = j-1$, (6.5) is

$$g_j^{(j-1)}(x) + c_{j,j-1} g_j^{(j-2)}(x) + \cdots + c_{j,1} g_j^{(1)}(x) = (j-1)!(-e^x)^{j-1}. \tag{6.8}$$

Since $g_j^{(s)}(0) = 0$ for $1 \le s \le j-2$, we conclude that $(j-1)!$ divides $g_j^{(j-1)}(0)$. By induction, we find that $g_j^{(s)}(0)$ are all divisible by $(j-1)!$ for $s \ge j-1$. In other words, for $j \le \ell+1$, $(j-1)!$ divides $C_{j,\ell}$. Since $j|(j-1)!$ if and only if j is a composite number and $j \ne 4$, we conclude that the fractional part of B_ℓ must be from the term $j = 4$ or $j = p$ where p is a prime.

We now consider $j = 4$. In this case,

$$g_4(x) = (1-e^x)^3 = 1 - 3e^x + 3e^{2x} - e^{3x},$$

and

$$g_4^{(\ell)}(x) = -3e^x + 3 \cdot 2^\ell e^{2x} - 3^\ell e^{3x}.$$

For even integers $\ell \geq 4$,

$$g_4^{(\ell)}(0) = -3 + 3 \cdot 2^\ell - 3^\ell \equiv 0 \pmod 4.$$

Therefore, $4|C_4$ and 4 does not divide the denominator of B_ℓ for even integer $\ell \geq 4$.

We have seen that

$$j|C_{j,\ell} \tag{6.9}$$

when j is composite. We are now left with the case $j = p$ where p is an odd prime. First, observe that

$$(1-v)^{p-1} = \frac{1-v^p}{1-v} + \sum_{j=1}^{(p-1)/2} \binom{p}{j} (-1)^j v^j \frac{1-v^{p-2j}}{1-v}.$$

Replacing v by e^x, we conclude that

$$\begin{aligned} g_p(x) &= (1-e^x)^{p-1} \\ &= \frac{(1-e^x)^p}{1-e^x} = \frac{(1-e^{px})}{1-e^x} + \sum_{j=1}^{(p-1)/2} \binom{p}{j} (-1)^j e^{jx} \frac{1-e^{(p-2j)x}}{1-e^x} \\ &= \sum_{j=0}^{p-1} e^{jx} + p \sum_{j=1}^{(p-1)/2} \frac{(p-1)!}{j!(p-j)!} (-1)^j e^{jx} \sum_{\nu=0}^{p-2j-1} e^{\nu x}. \end{aligned}$$

This implies that

$$g_p^{(\ell)}(0) \equiv \sum_{j=0}^{p-1} j^\ell \pmod p.$$

Suppose ω_p is a generator for $\mathbf{F}_p^*$. If $p-1$ divides ℓ, then

$$g_p^{(\ell)}(0) = \sum_{t=1}^{p-1} t^\ell \equiv \sum_{t=1}^{p-1} \omega_p^{\ell t} \equiv \sum_{t=1}^{p-1} 1 = p-1 \equiv -1 \pmod p.$$

If $p-1 \nmid \ell$, then

$$g_p^{(\ell)}(0) = \sum_{t=1}^{p-1} \omega_p^{\ell t} = \frac{1-\omega_p^{\ell(p-1)}}{1-\omega_p^\ell} \equiv 0 \pmod p.$$

Therefore, for odd prime p,

$$g_p^{(\ell)}(0) \equiv \begin{cases} -1 & \pmod{p} \quad \text{if } (p-1)|\ell, \\ 0 & \pmod{p} \quad \text{if } (p-1) \nmid \ell. \end{cases} \tag{6.10}$$

Combining (6.3), (6.9) and (6.10), we conclude that

$$B_\ell + \frac{1}{2} + \sum_{\substack{(p-1)|\ell \\ p \text{ an odd prime}}} \frac{1}{p} = B_\ell + \sum_{\substack{(p-1)|\ell \\ p \text{ a prime}}} \frac{1}{p}$$

is an integer.

7. Concluding remarks

In [4, Sect. 16], Glaisher gave a proof of (2.10) using essentially a combinatorial argument. Viewing $a_{p-1,2r+1}$ as a sum of products of distinct $2r+1$ integers chosen from the set $\{1, 2, \ldots, p-1\}$, he decomposed it into smaller sums according to the number of chosen integers a whose complementary factors (i.e., $p-a$) were not simultaneously chosen. These smaller sums then appear with powers of p as factors. Building on his combinatorial arguments, in [4, Sect. 19], he also showed that for $p > 5$ and $r \geq 2$,

$$a_{p-1,2r+1} \equiv \frac{p(p-2r-1)}{2} a_{p-1,2r} \pmod{p^4}. \tag{7.1}$$

In a follow up paper [5], he gave the congruences (2.8) and (2.9). Although Glaisher's proofs are elementary, they appear to have never been included in standard number theory textbooks, possibly because the combinatorial nature of these proofs does not align with the typical techniques covered in such texts.

Our final question is whether we could derive (7.1) using the congruences we established in this short note. The answer is probably no. However, using what we have discussed in this article, we could show a weaker version of (7.1) which is, for $r \geq 1$,

$$a_{p-1,2r+1} \equiv -\frac{2r+1}{2} p a_{p-1,2r} \pmod{p^3}. \tag{7.2}$$

To prove (7.2), multiply (2.8) throughout by $-(2r+1)p/2$ to deduce that

$$-\frac{2r+1}{2} p a_{p-1,2r} \equiv p^2 \frac{2r+1}{4r} B_{2r} \pmod{p^3}. \tag{7.3}$$

Since the right-hand sides of (7.3) and (2.9) are identical, the left-hand sides of (7.3) and (7.2) are congruent modulo p^3 and the proof of (7.2) is complete.

Multiplying both sides of (3.5) by $(2r+1)p/2$, we obtain, for $r \geq 2$,

$$\frac{2r+1}{2} p s_{2r} \equiv \frac{2r+1}{2} p^2 B_{2r} \pmod{p^4}. \tag{7.4}$$

Comparing that with (3.6), we obtain a congruence for s_k given by

$$s_{2r+1} \equiv \frac{2r+1}{2} p s_{2r} \pmod{p^4}$$

valid for $r \geq 2$. This may be viewed as "an analogue of Glaisher's congruence" for the sequence s_k.

Finally, from (2.8) and (3.5), we obtain, for $r \geq 1$,

$$-2r a_{p-1,2r} \equiv s_{2r} \pmod{p^2}. \tag{7.5}$$

Similarly, from (2.9) and (3.6), we deduce that, for $r \geq 1$,

$$2r a_{p-1,2r+1} \equiv s_{2r+1} \pmod{p^3}. \tag{7.6}$$

Congruences (7.5) and (7.6) in its weaker form $\pmod{p^2}$ can be found in an article of Z.H. Sun [7, p. 212, line 3].

We end this article by asking for direct proofs of the interesting congruences (7.5) and (7.6) without using the intermediate congruences involving the Bernoulli numbers.

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FIVE PROOFS THAT SIN n^2 DOES NOT CONVERGE

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Dedicated to the memory of Professor Michael von Renteln (1942-2026)

Abstract. In this survey we present five proofs that the sequence $\sin n^2$ is not convergent: three elementary ones, one using Kronecker's denseness theorem in ergodic theory according to which e^{in} is dense in the unit circle, and the standard one based on Weyl's theory on equidistributions modulo 1.

1. Introduction

A standard ingredient in various undergraduate courses in Mathematics is to ask our students whether the sequence $\sin n$ converges. One of the easiest proofs goes as follows: if $\sin n \to L$, then

$$2\cos n \sin 1 = \sin(n+1) - \sin(n-1) \to 0,$$

and so $\cos n \to 0$. In particular $\cos(2n) \to 0$. Moroever,

$$\sin(2n) = 2 \sin n \cos n \to 0,$$

contradicting the fact that $\cos^2 x + \sin^2 x = 1$.

On the other hand, we were not aware of a similar proof for $\sin n^2$. In this note we provide five proofs of the divergence, with increasing order of non-elementarity. The fact itself could have been familiar e.g. to Carl Friedrich Gauß (1777-1855), Leopold Kronecker (1823-1891), but surely to Maurice Fréchet (1878-1973). Since the times of Hermann Weyl (1895-1955), the starting point of the theory of uniform distribution modulo 1, the non-existence of such limits are trivial consequences of the much more

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general property that for irrational ξ the sequence $n^p\xi - \lfloor n^p\xi \rfloor$ is uniformly distributed ($p = 1, 2, 3\dots$). In our viewpoint it is always interesting though to find individual proofs of this weaker properties which are accessible to a broad audience, in particular to all students beginning their course in Real Analysis.

So here is the main fact which we are going to analyze. Let $\mathbb{N} = \{0, 1, 2\dots\}$.

Proposition 1.1. *The sequence* $(\sin n^2)$ *does not converge.*

2. First elementary proof

This proof, due to Gérald Tenenbaum, is based twice on a proof by contradiction.

Proof. Suppose that $L := \lim_{n\to\infty} \sin n^2$ exists.
Case 1. $L = 0$. Since

$$\sin(2n+1) = \sin\left((n+1)^2 - n^2\right) = \sin(n+1)^2 \cos n^2 - \sin n^2 \cos(n+1)^2,$$

we deduce that $\sin(2n+1) \to 0$. But the identity

$$\sin x - \sin y = 2\cos\tfrac{1}{2}(x+y)\ \sin\tfrac{1}{2}(x-y)$$

implies $2\sin 1\cos(2n) = \sin(2n+1) - \sin(2n-1) \to 0$, and hence $\cos(2n) \to 0$. Since

$$\sin(2n)\cos 1 + \cos(2n)\sin 1 = \sin(2n+1) \to 0,$$

we conclude that $\sin(2n) \to 0$. This is a contradiction since $\sin^2(2n) + \cos^2(2n) = 1$.

Case 2. $L \neq 0$. We first show that $\cos n^2$ converges, too (note that trivially $|\cos n^2|$ converges to $\sqrt{1-L^2}$). In fact, the hypothesis $\sin n^2 \to L \neq 0$ and the identity

$$\sin(4n^2) = 2\sin(2n^2)\cos(2n^2) = 4\sin n^2 \cos n^2 \cos(2n^2)$$

imply that

$$\cos n^2 \cos(2n^2) \to 1/4. \tag{2.1}$$

Moreover,

$$\cos(2n^2) = (\cos n^2)^2 - (\sin n^2)^2 = 1 - 2(\sin n^2)^2 \to 1 - 2L^2.$$

Hence, due to (2.1), $1-2L^2 \neq 0$, and so $\cos n^2$ converges, say $\cos n^2 \to M$. Moreover, by (2.1) again, $M \neq 0$. Next we use that

$$\sin(n+1)^2 = \sin n^2 \cos(2n+1) + \cos n^2 \sin(2n+1), \tag{2.2}$$

or equivalently

$$\begin{aligned}\sin(n+1)^2 - \sin n^2 = &\quad \sin n^2\big(\cos(2n+1)-1\big) + \cos n^2 \sin(2n+1)\\ = &\ (\sin n^2 - L)(\cos(2n+1)-1) + L(\cos(2n+1)-1)\\ &\quad +\big[(\cos n^2 - M)\sin(2n+1) + M\sin(2n+1)\big].\end{aligned}$$

Passing to the limit, we deduce that

$$L\big(\cos(2n+1)-1\big) + M\sin(2n+1) \to 0.$$

Hence, by using the formula $\cos(2x) = 1-2\sin^2 x$ again,

$$2\big[L\sin(n+\tfrac{1}{2}) - M\cos(n+\tfrac{1}{2})\big]\sin(n+\tfrac{1}{2}) \to 0. \tag{2.3}$$

In order to be able to conclude that the content in the brackets tends to zero, we need to convince us that, under our hypothesis $\sin n^2 \to L$, the bounded sequence $|\sin\big(n+\frac{1}{2}\big)|$ is also bounded away from zero, or equivalently, that there is no subsequence n_k with $\sin(n_k+1/2) \to 0$. To see this, suppose the contrary that $\sin(n_k+1/2)\to 0$. Then $\cos(n_k+1/2)$ does not converge to 0. Hence there is a subsequence n_{k_s} such that $C := \lim_s \cos(n_{k_s}+1/2)$ exists and $C \neq 0$. Thus, for every $j \in \mathbb{Z}\setminus\{0\}$, the identity

$$\sin(n_{k_s}+1/2+j) = \sin(n_{k_s}+1/2)\cos j + \cos(n_{k_s}+1/2)\sin j$$

implies that $L_j := \lim_s \sin(n_{k_s}+1/2+j)$ exists and $L_j = C\sin j \neq 0$.

Replacing in (2.3) n by $n+j$, we deduce from this that $\cot(n_{k_s}+\frac{3}{2})$ and $\cot(n_{k_s}+\frac{5}{2})$ converge to the same limit $t := L/M$. However this is impossible in view of the formula

$$\cot(x+1) = \frac{\cot x\cot 1 - 1}{\cot x + \cot 1} \qquad \big(x+1 \not\equiv 0 \pmod{\pi}\big), \tag{2.4}$$

applied to $x = n_{k_s}+3/2$. In fact, by passing to the limit $s\to\infty$, we infer that $t \neq -\cot 1$ and

$$t(t+\cot 1) = t\cot 1 - 1.$$

Hence $t^2 = -1$, an obvious contradiction.

Thus we may assume that $\sin(n+\frac{1}{2})$ is bounded away from 0. By (2.3), we get that both $\cot(n+\frac{1}{2})$ and $\cot(n-\frac{1}{2})$ tend to $t = L/M$, which again leads to a contradiction in view of (2.4). □

3. Second elementary proof

Our second proof is based on the theory of cluster points: a bounded sequence (x_n) is convergent if and only if (x_n) has a single cluster point (where as usual a is a cluster point if and only if there is a subsequence (x_{n_k}) converging to a).

Proof. Again suppose that $\sin n^2 \to L \in [-1,1]$. Then $|\cos n^2| \to \sqrt{1-L^2}$. Now using that $(n+1)^2 = n^2 + (2n+1)$, we get

$$\sin(n+1)^2 = \sin n^2 \cos(2n+1) + \cos n^2 \sin(2n+1). \tag{3.1}$$

Let B denote the set of cluster points of the sequence $(\cos(2n+1))$. We shall prove that, under the hypothesis $\sin n^2 \to L$, $\operatorname{card} B = 1$ (that is, B is a singleton). So let $a \in B$. Then, due to (3.1),

$$L^2(1-a)^2 = (1-L^2)(1-a^2). \tag{3.2}$$

Case 1. $L^2 \neq 1$. Then either $a = 1$ or

$$\frac{L^2}{1-L^2} = \frac{1+a}{1-a} \in [0,\infty[. \tag{3.3}$$

Since the right-hand side is strictly increasing as a function of a whenever $-1 \leqslant a < 1$, we deduce that, for fixed L with $L^2 \neq 1$, there is a unique solution $a_L \in [-1,1[$ to (3.3). We conclude that $B \subseteq \{1, a_L\}$.

Case 2. $L^2 = 1$. Then $a = 1$ is the only solution to (3.2), so $B = \{1\}$.

We now show that $1 \in B \Rightarrow L^2 = 1$, which plainly implies the desired conclusion that $\operatorname{card} B = 1$. Let $\cos(2n_k+1) \to 1$. Then $\sin(2n_k+1) \to 0$. Using for $j = 3, 5$ the identity

$$\begin{aligned}\sin(2n_k+j) &= \sin((2n_k+1)+(j-1))\\ &= \sin(2n_k+1)\cos(j-1) + \cos(2n_k+1)\sin(j-1),\end{aligned}$$

we see that $\lim \sin(2n_k+3) = \sin 2$ and $\lim \sin(2n_k+5) = \sin 4$. Passing, if necessary, to a subsequence, this produces two distinct elements of B, a_1 and a_2, both different from 1 ($a_1 \in \{\pm\sqrt{1-\sin^2 2}\}, a_2 \in \{\pm\sqrt{1-\sin^2 4}\}$). If now $L^2 \neq 1$, we are in case 1 and (3.3) holds with $a = a_1$ and $a = a_2$, so $a_1 = a_2$, a contradiction. We conclude that $L^2 = 1$.

Therefore, we have proved that $\operatorname{card} B = 1$, say $B = \{a\}$. However, this is not possible. Indeed, using twice

$$\cos x - \cos y = -2\sin\tfrac{1}{2}(x+y)\ \sin\tfrac{1}{2}(x-y),$$

we obtain

$$\cos(2n+3) - \cos(2n-1) = -2\sin(2n+1)\sin 2,$$

implying $\sin(2n+1) \to 0$, and

$$\cos(2n+1) - \cos(2n-1) = -2\sin(2n)\sin 1,$$

implying $\sin(2n) \to 0$. Thus $\sin n$ would be convergent. As we have seen in the introduction, this is not the case. □

4. Third proof: based on Kronecker's density theorem in ergodic theory.

Here we use Kronecker's result that $\{e^{in\theta} : n \in \mathbb{N}\}$ is dense in the unit circle $\mathbb{T}$ for every $\theta \in \mathbb{R}$ with $\theta/2\pi \notin \mathbb{Q}$. For the standard proof, see e.g. [5, p. 1878]. This makes our proof non-elementary, as it uses that π is irrational.

Lemma 4.1. *$\sin(2n+1)$ and $\cos(2n+1)$ are dense in $[-1,1]$.*

Proof. This is a direct consequence of Kronecker's result. To see this, it suffices to write

$$e^{i(2n+1)\theta} = e^{i\theta}\ e^{in(2\theta)}.$$

Since θ/π is irrational, too, $e^{in(2\theta)}$ is dense in $\mathbb{T}$ and so does $e^{i(2n+1)\theta}$. Take $\theta = 1$. If $t_0 \in [-1,1]$, put $z := e^{i\arccos t_0}$, respectively $w := e^{i\arcsin t_0}$. Now choose subsequences z_k of $e^{i(2n+1)}$ converging to z, respectively w_k converging to w. Taking real (rep. imaginary) parts yields the assertion: $\operatorname{Re} z_k \to t_0$ and $\operatorname{Im} w_k \to t_0$. □

We are now ready to give our third proof of Proposition 1.1.

Proof. Suppose that $\sin n^2 \to L$. Since $\big(\cos(2n+1)\big)$ is dense in $[-1,1]$, we may choose a subsequence (n_k) such that $\cos(2n_k+1) \to -1$. Hence $\sin(2n_k+1) \to 0$. Thus, due to (3.1), $L = -L$, and so $L = 0$.

Moreover, as $\big(\sin(2n+1)\big)$ is dense in $[-1,1]$, too, there exists (n_k) such that $\sin(2n_k+1) \to 1$. Since $L = 0$, we deduce from (3.1) that

$\cos n_k^2 \sin(2n_k+1) \to 0$, and so, $\cos n_k^2 \to 0$. But this leads to a contradiction since $(\cos n_k^2)^2 \to 1 - L^2 = 1$. $\square$

5. Fourth proof: Based on Weyl's theorem in ergodic theory

In this final, non elementary, approach we need the notion of equidistribution (or uniform distribution). Here a sequence (x_n) of real numbers is said to be *uniformly distributed modulo 1* if for every $0 \leqslant a < b \leqslant 1$

$$\lim_{N\to\infty} \frac{1}{N} \text{card}\Big\{n \in \{1, \dots, N\} : x_n - \lfloor x_n \rfloor \in [a, b]\Big\} = b - a.$$

It is evident that $\big(x_n - \lfloor x_n \rfloor\big)$ is dense in $[0, 1]$ for any sequence (x_n) uniformly distributed modulo 1.

Kronecker's theorem implies that $n\xi - \lfloor n\xi \rfloor$ is dense in $[-1, 1]$ for irrational ξ. But much more holds. Theorem 3.2 in [2, p. 27], due to Weyl, tells us that if

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_d x^d$$

is any non-constant polynomial in $\mathbb{R}[x]$ with at least one of the coefficients a_j with $j > 0$ irrational, then $(p(n))$ is uniformly distributed modulo 1. This holds in particular for the polynomial $p(x) = \xi x^2$, $\xi \in \mathbb{R}$ irrational. Hence $n^2\xi - \lfloor n^2\xi \rfloor$ is dense in $[0, 1]$. Consequently $2\pi(n^2\xi - \lfloor n^2\xi \rfloor)$ is dense in $[0, 2\pi]$ and so $e^{2\pi i n^2 \xi}$ is dense on the unit circle. This finally implies that with $\xi = 1/2\pi$, $\sin(n^2)$ is dense in $[-1, 1]$. In particular, $\sin n^2$ does not converge.

6. A short elementary, but tricky proof

Here we present a proof which was motivated (after we had finished the first draft of our note) by an erroneous attempt we found on MSE [7].

Suppose that $\sin n^2 \to L$. If $L = 0$, then we get a contradiction (see the first proof). If $L \neq 0$, then we showed in our first proof that $\cos n^2$ converges too, say $\cos n^2 \to M$. Now we work with the triple $(3, 4, 5)$ satisfying $3^2 + 4^2 = 5^2$. Hence

$$\begin{aligned} L \longleftarrow \sin(5^2n^2) &= \sin(4^2n^2 + 3^2n^2) = \sin(16n^2)\cos(9n^2) + \cos(16n^2)\sin(9n^2) \\ &\to 2ML \\ M \longleftarrow \cos(5^2n^2) &= \cos(4^2n^2 + 3^2n^2) = \cos(16n^2)\cos(9n^2) - \sin(16n^2)\sin(9n^2) \\ &\to M^2 - L^2. \end{aligned}$$

Consequently

$$L = 2ML, \quad M = M^2 - L^2, \quad M^2 + L^2 = 1.$$

Since $L \neq 0$, $M = 1/2$. Hence $1/2 = 1/4 - L^2$, which has no solution. Thus the case where $L \neq 0$, does not appear. □

Remark Here we would like to give the following homework for our students: *derive similar proofs for* $\cos n^2$.

7. Beyond the polynomial x^2

A natural question arises whether such elementary proofs can also be given for $\sin n^p$ with $p \in \mathbb{N}$, $p \geqslant 3$. Note that due to Weyl's theorem, the (non-elementary) fourth proof immediately carries over to this case (take $p(x) = \frac{1}{2\pi}x^p$). The other proofs here do not seem to be extendable to the general case. In an upcoming research paper [6], we succeeded though, by using Chebyshev polynomials. There we also characterized several classes of quadratic polynomials $p(x)$ for which $\sin p(n)$ is convergent. For instance, one may take

$$\sin\Big(c_0 + \frac{2\pi}{3}a_1 n + \frac{2\pi}{3}a_2 n^2\Big),$$

where 3 divides a_1 but not a_2 and where

$$c_0 = c_0(p) := \pi\Big(\frac{p}{2} - \frac{1}{3}a_2\Big)$$

for some odd integer p. This is very amusing, we think. On the other hand, many questions within this subject remain open. E.g. will it ever be possible to describe all those polynomials p for which $\sin p(n)$ converges? We think that it surely needs advanced methods to tackle this kind of problems. One of the famous unsolved problem for instance is: what are the limit points of the sequence $n \sin n$? This depends on the unknown irrationality exponent of π and is still very mysterious. By additionally using Diophantine approximation and this notion of irrationality exponent one is able to study the cluster sets of sequences of type $n^s(\sin(\alpha n))^{n^r}$ and much more. See e.g. [1], [3], [4]. For instance, it is known that the cluster set C_r of

$$\Big(1 - \sin^6(\pi\sqrt{2}\, n)\Big)^{n^r}$$

equals $[0, 1]$ whenever $0 < r < 6$, that C_6 is strictly smaller than $[0, \rho]$ for some $\rho < 1$, and that $C_r = \{0\}$ for $r > 6$.

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MONOTONICITY PROPERTIES OF NORMALIZED POWER INTEGRALS

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Abstract. In this paper, we establish monotonicity and strict log-convexity properties of the normalized power integral functional

$$\Phi(t) := \frac{\int_a^b f(x)^t\,dx}{\left(\int_a^b f(x)\,dx\right)^t},$$

where f is a continuous, strictly increasing, positive function on $[a,b]$. As a consequence, we derive a refined inequality of Qi-type valid for exponents greater than or equal to one.

1. Introduction

Integral inequalities involving power functions and their normalized moments are fundamental tools in analysis, with wide-ranging applications in functional analysis, probability theory, and mathematical physics. A classical problem is to understand how the L^p-norms or moments of a positive function vary as the exponent p changes, which leads to inequalities comparing these moments.

Monotonicity properties of normalized power integrals have been studied extensively. For instance, the well-known monotonicity of L^p-norms for $1 \leq p < q$, i.e.,

$$\|f\|_p \leq \|f\|_q,$$

is a classical result often proved using Hölder's inequality or Jensen's inequality (see e.g. [1, 2]). This monotonic behavior is closely linked to the convexity or log-convexity properties of moment generating functionals associated with the underlying function.

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More recently, researchers have focused on exploiting log-convexity to derive sharper and more general inequalities. Qi and collaborators introduced several integral inequalities involving power means and moments, often improving or extending classical results [3, 4]. The study of log-convexity of normalized moment functionals allows one to unify and strengthen such inequalities.

In this work, we focus on the functional

$$\Phi(t) := \frac{\int_a^b f(x)^t \, dx}{\left(\int_a^b f(x) \, dx\right)^t},$$

where $f : [a, b] \to (0, \infty)$ is continuous and strictly increasing. We prove that Φ is strictly log-convex on $(0, \infty)$. This key property yields monotonicity results for Φ, leading to new integral inequalities that generalize and refine classical moment inequalities, including improved Qi-type inequalities valid for exponents greater than or equal to one.

Our approach builds on the foundational work on moment inequalities and log-convexity [5, 6], and offers a streamlined proof technique along with explicit inequalities that may be of independent interest in the theory of integral inequalities and their applications.

2. Preliminaries

In this section, we recall some basic definitions and properties of convex functions which are essential for the development of our main results. These fundamental concepts are provided for the sake of completeness and clarity [5].

Definition 2.1. A function $g : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality

$$g((1 - \lambda)x + \lambda y) \leq (1 - \lambda)g(x) + \lambda g(y) \tag{2.1}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. If the inequality is strict for $x \neq y$ and $\lambda \in (0, 1)$, then g is called strictly convex.

Remark 2.2. If g is twice differentiable on an open interval I, then g is convex on I if and only if $g''(x) \geq 0$ for all $x \in I$. Furthermore, if $g''(x) > 0$ for all $x \in I$, then g is strictly convex on I.

Definition 2.3. A positive function $g : I \to (0, \infty)$ is called log-convex if its logarithm $\ln g$ is a convex function.

3. Main Result

In this section, we present our main results. We study the normalized power integral functional for continuous and strictly increasing functions. Using its strict log-convexity, we derive new inequalities that generalize classical moment inequalities.

Theorem 3.1. *Let $f : [a, b] \to (0, \infty)$ be continuous and strictly increasing. Define, for $t > 0$,*

$$\Phi(t) := \frac{\displaystyle\int_a^b f(x)^t \, dx}{\left(\displaystyle\int_a^b f(x) \, dx\right)^t}.$$

Then the function Φ is strictly log-convex on $(0, \infty)$. Moreover, for any α, β satisfying

$$\alpha > \beta \geq 1,$$

the following inequality holds:

$$\frac{\int_a^b f(x)^\alpha \, dx}{\left(\int_a^b f(x) \, dx\right)^\alpha} > \frac{\int_a^b f(x)^\beta \, dx}{\left(\int_a^b f(x) \, dx\right)^\beta}.$$

Proof. Let

$$F(t) := \int_a^b f(x)^t \, dx, \qquad L := \int_a^b f(x) \, dx > 0,$$

and define

$$G(t) := \ln \Phi(t) = \ln F(t) - t \ln L.$$

First we compute the derivative of G. Since

$$F'(t) = \int_a^b f(x)^t \ln f(x) \, dx,$$

it follows that

$$G'(t) = \frac{F'(t)}{F(t)} - \ln L = \frac{\int_a^b f(x)^t \ln f(x) \, dx}{\int_a^b f(x)^t \, dx} - \ln L.$$

Differentiating once more, we obtain

$$G''(t) = \frac{\int_a^b f(x)^t (\ln f(x))^2 \, dx}{\int_a^b f(x)^t \, dx} - \left(\frac{\int_a^b f(x)^t \ln f(x) \, dx}{\int_a^b f(x)^t \, dx} \right)^2 .$$

By the Cauchy-Schwarz inequality for integrals we have

$$\left(\int_a^b f(x)^t \ln f(x) \, dx \right)^2 \leq \left(\int_a^b f(x)^t \, dx \right) \left(\int_a^b f(x)^t (\ln f(x))^2 \, dx \right).$$

Consequently,

$$G''(t) \geq 0.$$

Moreover, equality occurs only when $\ln f(x)$ is constant almost everywhere on $[a, b]$. Since f is strictly increasing, $\ln f(x)$ cannot be constant, and therefore

$$G''(t) > 0 \qquad \text{for all } t > 0.$$

Thus G is strictly convex on $(0, \infty)$, which implies that $\Phi(t) = e^{G(t)}$ is strictly log-convex.

Next we examine the behavior of G at $t = 1$. Since $F(1) = L$, we have

$$G(1) = \ln F(1) - \ln L = 0.$$

To determine the sign of $G'(1)$, define the probability measure

$$d\mu(x) := \frac{f(x)}{L} \, dx, \qquad L = \int_a^b f(x) \, dx.$$

Then

$$G'(1) = \int_a^b \ln f(x) \, d\mu(x) - \ln\left(\int_a^b f(x) \, d\mu(x) \right).$$

Since ln is strictly concave and f is strictly increasing (hence non-constant), Jensen's inequality gives

$$\int_a^b \ln f(x) \, d\mu(x) < \ln\left(\int_a^b f(x) \, d\mu(x) \right),$$

which immediately implies

$$G'(1) > 0.$$

Since $G''(t) > 0$, the derivative $G'(t)$ is strictly increasing. Because $G'(1) > 0$, we conclude that

$$G'(t) > 0 \qquad \text{for all } t \geq 1,$$

so that G is strictly increasing on $[1, \infty)$. Consequently,

$$\Phi(t) = e^{G(t)}$$

is also strictly increasing on this interval.

Therefore, for any $\alpha > \beta \geq 1$ we obtain

$$\Phi(\alpha) > \Phi(\beta),$$

that is,

$$\frac{\int_a^b f(x)^\alpha \, dx}{\left(\int_a^b f(x)\, dx\right)^\alpha} > \frac{\int_a^b f(x)^\beta \, dx}{\left(\int_a^b f(x)\, dx\right)^\beta}.$$

This completes the proof. □

Remark 3.2. Based on Theorem 3.1, the following observations can be made concerning the behavior of the function Φ:

(I) The condition $\alpha > \beta \geq 1$ is essential for the inequality

$$\frac{\int_a^b f(x)^\alpha \, dx}{\left(\int_a^b f(x)\, dx\right)^\alpha} > \frac{\int_a^b f(x)^\beta \, dx}{\left(\int_a^b f(x)\, dx\right)^\beta}$$

to hold, since Theorem 3.1 establishes that Φ is strictly increasing on the interval $[1, \infty)$.

(II) In contrast, when $0 < \beta < \alpha < 1$, the function Φ is strictly decreasing on $(0, 1)$, as implied by the strict convexity of $\log \Phi$. Therefore, the inequality in Theorem 3.1 reverses:

$$\frac{\int_a^b f(x)^\alpha \, dx}{\left(\int_a^b f(x)\, dx\right)^\alpha} < \frac{\int_a^b f(x)^\beta \, dx}{\left(\int_a^b f(x)\, dx\right)^\beta}.$$

(III) For the mixed case $\beta < 1 < \alpha$, the monotonicity of Φ changes at $t = 1$, which is the transition point between decreasing and increasing behavior. Thus, the inequality cannot be concluded directly from Theorem 3.1, and requires a more refined analysis depending on the specific values of α and β.

Corollary 3.3. *Let $f : [a, b] \to (0, \infty)$ be continuous and not constant. Then for any $0 < p < q$, the following strict inequality holds:*

$$\|f\|_p < \|f\|_q,$$

where

$$\|f\|_r := \left(\frac{1}{b-a}\int_a^b f(x)^r \, dx\right)^{1/r}$$

is the normalized L^r-norm of f on $[a, b]$.

Proof. Define the function

$$\psi(r) := \ln\left(\int_a^b f(x)^r\,dx\right), \quad r > 0.$$

Since $f(x) > 0$ and continuous, the map $r \mapsto f(x)^r$ is log-convex for each x, and integration preserves log-convexity. Hence, ψ is strictly convex on $(0,\infty)$.

Define

$$\phi(r) := \frac{1}{r}\psi(r) = \frac{1}{r}\ln\left(\int_a^b f(x)^r\,dx\right).$$

Note that $\phi(r) = \ln\|f\|_r + \frac{1}{r}\ln(b-a)$, so ϕ and $\ln\|f\|_r$ have the same monotonicity behavior. Differentiating ϕ, we obtain:

$$\phi'(r) = \frac{r\psi'(r) - \psi(r)}{r^2}.$$

Since ψ is strictly convex, the inequality

$$\psi'(r) > \frac{\psi(r)}{r}$$

holds for all $r > 0$, which implies $\phi'(r) > 0$. Thus, ϕ is strictly increasing, and so is $\|f\|_r$. Therefore, for $0 < p < q$ we have

$$\|f\|_p < \|f\|_q.$$

Strictness follows from the strict convexity of ψ, which holds if f is not constant almost everywhere. □

Theorem 3.4 (Interpolated Power Moment Inequality)**.** *Let $f : [a,b] \to (0,\infty)$ be continuous and strictly increasing, and let $\alpha, \beta > 0$ with $\alpha \neq \beta$. Then, for any $\theta \in (0,1)$, the following inequality holds:*

$$\frac{\int_a^b f(x)^{\theta\alpha+(1-\theta)\beta}\,dx}{\left(\int_a^b f(x)\,dx\right)^{\theta\alpha+(1-\theta)\beta}} \le \left(\frac{\int_a^b f(x)^\alpha\,dx}{\left(\int_a^b f(x)\,dx\right)^\alpha}\right)^\theta \left(\frac{\int_a^b f(x)^\beta\,dx}{\left(\int_a^b f(x)\,dx\right)^\beta}\right)^{1-\theta}.$$

Proof. Define the normalized power integral functional

$$\Phi(t) := \frac{\int_a^b f(x)^t\,dx}{\left(\int_a^b f(x)\,dx\right)^t}, \quad t > 0.$$

From Theorem 3.1, we know that Φ is strictly log-convex on $(0,\infty)$.

By the definition of log-convexity, for any $t_1, t_2 > 0$ and $\theta \in (0,1)$, we have

$$\log \Phi(\theta t_1 + (1-\theta)t_2) \leq \theta \log \Phi(t_1) + (1-\theta) \log \Phi(t_2).$$

Exponentiating both sides gives

$$\Phi(\theta t_1 + (1-\theta)t_2) \leq [\Phi(t_1)]^{\theta} [\Phi(t_2)]^{1-\theta}.$$

Finally, by choosing $t_1 = \alpha$ and $t_2 = \beta$, we obtain exactly the inequality stated in the theorem:

$$\frac{\int_a^b f(x)^{\theta\alpha+(1-\theta)\beta} dx}{\left(\int_a^b f(x)dx\right)^{\theta\alpha+(1-\theta)\beta}} \leq \left(\frac{\int_a^b f(x)^{\alpha} dx}{\left(\int_a^b f(x)dx\right)^{\alpha}}\right)^{\theta} \left(\frac{\int_a^b f(x)^{\beta} dx}{\left(\int_a^b f(x)dx\right)^{\beta}}\right)^{1-\theta}.$$

This completes the proof. □

Remark 3.5. Since Φ is strictly log-convex, the inequality is actually strict ($<$) whenever $\alpha \neq \beta$ and $\theta \in (0,1)$, even though it is conventionally stated in the non-strict form ($\leq$) in the Theorem 3.4.

Corollary 3.6 (Special Case of Theorem 3.4 with $\alpha = 2$, $\beta = 1$)**.** *Let $f : [a,b] \to (0,\infty)$ be a continuous and strictly increasing function. Then, for any $\theta \in (0,1)$, the following inequality holds:*

$$\frac{\int_a^b f(x)^{\theta+1}\, dx}{\left(\int_a^b f(x)\, dx\right)^{\theta+1}} \leq \left(\frac{\int_a^b f(x)^2\, dx}{\left(\int_a^b f(x)\, dx\right)^2}\right)^{\theta}.$$

Remark 3.7. Setting $\theta = \frac{1}{2}$ in Corollary 3.6 yields the inequality

$$\left(\int_a^b f(x)^{\frac{3}{2}}\, dx\right)^2 \leq \left(\int_a^b f(x)\, dx\right)\left(\int_a^b f(x)^2\, dx\right)$$

which is the Cauchy-Scwartz inequality.

Moreover, since f is strictly increasing and positive, the inequality is strict unless f is constant almost everywhere.

Remark 3.8. Consider the function $f(x) = x$ on the interval $[a,b]$ with $0 < a < b$. For $\theta = \frac{1}{2}$, Corollary 3.6 states the inequality

$$\left(\int_a^b f(x)^{\frac{3}{2}}\, dx\right)^2 \leq \left(\int_a^b f(x)\, dx\right)\left(\int_a^b f(x)^2\, dx\right).$$

Explicitly, after simplification, this is equivalent to

$$\frac{4}{25}\left(b^{\frac{5}{2}}-a^{\frac{5}{2}}\right)^2 \leq \frac{1}{6}\left(b^2-a^2\right)\left(b^3-a^3\right).$$

This inequality illustrates the interpolating power moment inequality for the simple strictly increasing function $f(x)=x$ and can be verified numerically for any $0<a<b$.

4. Conclusion

We proved that the normalized power moment functional Φ of a strictly increasing positive continuous function is strictly log-convex on $(0,\infty)$. This yields a new class of integral inequalities that strictly order normalized moments for exponents $\alpha>\beta\geq 1$.

Our result refines moment inequality theory and provides a foundation for further research in generalized moment inequalities and related integral operators. Future work may extend these findings to broader function classes and weighted settings.

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SURY'S IDENTITY AND THE F-ENVELOPE OF A FIBONACCI-LUCAS IDENTITY

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Abstract. The F-envelope measures the coverage level of a Fibonacci-Lucas identity in the set of all (a, b)-Fibonacci sequences. Sury's Fibonacci-Lucas identity is not an isolated identity in the set of (a, b)-Fibonacci sequences; its F-envelope is the set of k-Jacobsthal sequences.

1. Introduction: The F-envelope of a Fibonacci-Lucas identity

1.1. Fibonacci and Lucas sequences. The Fibonacci $F = \{F_n\}_{n\geq 0}$ and the Lucas $L = \{L_n\}_{n\geq 0}$ sequences are defined by the same recurrence relation

$$A_{n+2} = A_{n+1} + A_n$$

with initial values $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$, respectively.

In [5], it is shown that many of the properties of Fibonacci numbers can be stated for a special class of sequences: the (a, b)-Fibonacci sequences. For a and b real numbers, $\mathcal{R}(a, b)$ denotes the set of all sequences with initial values A_0 and A_1, and with the succeeding terms given by

$$A_{n+2} = aA_{n+1} + bA_n.$$

The set $\mathcal{R}(a, b)$ contains two distinguished elements: the (a, b)-Fibonacci sequence $\{F_n^{a,b}\}_{n\geq 0}$ with initial values $F_0^{a,b} = 0$, $F_1^{a,b} = 1$, and the (a, b)-Lucas sequence $\{L_n^{a,b}\}_{n\geq 0}$ with initial values $L_0^{a,b} = 2$, $L_1^{a,b} = a$.

The ordinary Fibonacci sequence $F = \{F_n\}_{n\geq 0}$ is the $(1, 1)$-Fibonacci sequence, and the ordinary Lucas sequence $L = \{L_n\}_{n\geq 0}$ is the $(1, 1)$-Lucas sequence. The $(1, 2)$-Fibonacci sequence is the Jacobsthal sequence $\{J_n\}_{n\geq 0}$, and the $(1, 2)$-Lucas sequence is the Jacobsthal-Lucas sequence

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$\{j_n\}_{n\geq 0}$. The k-Fibonacci sequence is the $(k,1)$-Fibonacci sequence, and the k-Jacobsthal sequence (in the sense of Falcon [4]) is the $(1,k)$-Fibonacci sequence.

1.2. The F-envelope of a Fibonacci-Lucas identity. For a given Fibonacci-Lucas identity $I(F,L)$ (for example, identity (1) below), the following problem naturally arises:

"Find the set of all (a,b)-Fibonacci sequences (together with the corresponding (a,b)-Lucas sequences) for which the identity $I(F,L)$ remains valid even if (F,L) is replaced by $(F^{a,b}, L^{a,b})$ (that is, F_n by $F_n^{a,b}$ and L_n by $L_n^{a,b}$)."

We call the set of all these (a,b)-Fibonacci sequences the *F-envelope* of the Fibonacci-Lucas identity. The F-envelope of an identity $I(F,L)$ is not empty since it contains the ordinary Fibonacci sequence F. The F-envelope measures the coverage level of the identity $I(F,L)$ in the set of all (a,b)-Fibonacci sequences.

1.3. Some examples. For a simple illustration of the above, we consider the following identity: $L_n^{a,b} = F_{n+1}^{a,b} + bF_{n-1}^{a,b}$ (see [5, Eq.(5)]). This means that the F-envelope of the Fibonacci-Lucas identity

$$L_n = F_{n+1} + F_{n-1}$$

is the set of k-Fibonacci sequences (i.e., the $(k,1)$-Fibonacci sequences).

A second Fibonacci-Lucas identity is the following:

$$L_n = F_n + 2F_{n-1}$$

It is straightforward to check that the F-envelope of this Fibonacci-Lucas identity is a singleton (there is no (a,b)-Fibonacci sequence different from the ordinary Fibonacci sequence F such that $L_n^{a,b} = F_n^{a,b} + 2F_{n-1}^{a,b}$ for every positive integer n). We call such an identity an *isolated identity* in the set of all (a,b)-Fibonacci sequences.

Now, using the Binet formulas for (a,b)-Fibonacci and (a,b)-Lucas numbers (see [5, Equations (8) and (9)]) it follows the following identity: $F_{2n}^{a,b} = F_n^{a,b}L_n^{a,b}$. So, the Fibonacci-Lucas identity

$$F_{2n} = F_nL_n$$

is a *universal identity* in the sense that the F-envelope of this identity is the set of all (a, b)-Fibonacci sequences.

2. The F-envelope of Sury's identity

2.1. Sury's identity. The noteworthy identity (1) below (see [13]),

$$2^{n+1}F_{n+1} = \sum_{i=0}^{n} 2^i L_i \qquad (1)$$

leads to numerous related identities and generalizations. Shortly after its proof in [13] (using a polynomial identity), new proofs and extensions of this identity were published in [3, 6, 7, 8] etc., and more recently in [1, 2, 11], etc. Our interest lies in the F-envelope of this Sury's Fibonacci-Lucas identity (1).

2.2. The F-envelope of (1). A closer examination of the short note [11] indicates the answer regarding the F-envelope of Sury's identity (1). First, in [11, Section 4], the following Jacobsthal analogue of (1),

$$2^{n+1}J_{n+1} = \sum_{i=0}^{n} 2^i j_i,$$

is proved using Euler's Telescoping Lemma. This means that the Jacobsthal sequence is an element of the F-envelope of Sury's identity. In [11, Section 5, Eq.(6)] is given the following generalization of (1) to (a, b)-Fibonacci numbers:

$$2^{n+1}F_{n+1}^{a,b} = \sum_{i=0}^{n} 2^i [L_i^{a,b} + (a-1)F_i^{a,b}].$$

Thus, the conclusion is immediate.

Theorem 2.1. *The F-envelope of Sury's identity is the set of k-Jacobsthal sequences.*

Although, the F-envelopes of the identities $L_n = F_{n+1} + F_{n-1}$ and (1) are different (F is the only common element), their coverage level for the two Fibonacci-Lucas identities in the set of all (a, b)-Fibonacci sequences is the same since there is a one-to-one correspondence between the two F-envelopes. Thus, Sury's Fibonacci-Lucas identity is not an isolated identity in the set of all (a, b)-Fibonacci sequences.

2.3. A corollary to Theorem 2.1. Since (1) is not an isolated identity in the set of (a,b)-Fibonacci sequences, Theorem 2.1 provides us with other examples of sequences with Sury's identity. Let's consider the following two examples:

Example 1 : $\{X_n\}_{n\geq 0} : 0, 1, 1, 1, 1, ..., 1, ...$ $\quad \{Y_n\}_{n\geq 0} : 2, 1, 1, 1, 1, ..., 1, ...$

Example 2 : (the periodic sequences with period 6 below)

$$\{X_n\}_{n\geq 0} : 0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, ...$$

(*see* A010892 *in the OEIS* [12])

$$\{Y_n\}_{n\geq 0} : 2, 1, -1-2, -1, 1, 2, 1, -1, -2, -1, 1, ...$$

for which Sury's identity,

$$2^{n+1}X_{n+1} = \sum_{i=0}^{n} 2^i Y_i,$$

holds.

The sequence A010892 above (called the Sastry sequence in [10] since it derives from Sastry's generalized Möbius function [9]), together with the sequence $0, 1, 1, ..., 1, ...$, are two distinguished elements (just like Fibonacci and Jacobsthal sequences) of the F-envelope of Sury's identity.

2.4. Remark. A similar investigation can be made for Fibonacci identities (without Lucas). The F-envelopes of the famous Cassini's, d'Ocagne's, Catalan's, Vajda's, and Honsberger's identities are the same, namely the set of k-Fibonacci sequences.

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PROBLEM SECTION

In volume 94(3-4) 2025 of the Mathematics Student, we had invited solutions for a set of five new problems, in addition to the problem 5 from 94(1-2). We have received solutions for problems 1 and 4 from 94(3-4). We shall present these solutions and the solution for problem 5 of 94(1-2) as still it remains unsolved. We shall give some more time for problems 2, 3 and 5 from 94(3-4).

We present five new problems in this volume and invite solutions for these problems from the readers till October 25, 2026. Correct solutions received by this date will be published in volume 95 (3-4) 2026 of The Mathematics Student, which is scheduled to be published in November 2025.
We sincerely acknowledge all the cooperation and support from the people who contribute problems and solutions to the problem section. We also welcome new problems along with detailed solutions for the problem section from the readers.

New Problems

MS 95(1-2) 2026 : Problem 1 (by **Dr. Nguyen Tien Lam**, High School for Gifted Students, Hanoi University of Science, Vietnam).

Find all pairs of positive integers (a, b) such that there exist infinitely many pairs of positive integers (m, n) for which both $m^3 + 3m^2 + an + b$ and $n^3 + 3n^2 + am + b$ are perfect cubes.

MS 95(1-2) 2026 : Problem 2 (by **Dr. Shpetim Rexhepi** and **Dr. Ilir Demiri**, Mother Teresa University, Skopje, North Macedonia).

Solve the following equation for $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $n \geq 3$.

$$\left(\frac{\log_n (n+1)^n}{n+1}\right)^x + \left(\frac{\log_n (n+1)^{2n}}{(\log_n (n+1)^n)^2 + 1}\right)^x = \frac{2}{\left(\log\left(1 + \frac{1}{(n+1)^{2n}}\right)\right)^x}$$

Problems 3-5 (by **Dr. B. Sury**, ISI, Bengaluru, India).

MS 95(1-2) 2026 : Problem 3.

Show that any monic polynomial f with real coefficients is the average $\dfrac{g+h}{2}$, where g, h are monic polynomials with all roots real.

MS 95(1-2) 2026 : Problem 4.

Let f be a monic polynomial of degree $n \in \mathbb{N}$ with real coefficients and suppose $a_0 > a_1 > \cdots > a_n$ are integers. Prove that $|f(a_i)| \geq \dfrac{n!}{2^n}$ for some $0 \leq i \leq n$.

MS 95(1-2) 2026 : Problem 5.

For $1 \leq k \leq n$, consider the set: equation

$$S_k(n) := \left\{ (a_1, a_2, \cdots, a_k) \in \mathbb{N}^k : \sum_{i=1}^{k} a_i = n \right\}.$$

For instance, $S_2(3) = \{(1,2),(2,1)\}$ and $S_2(4) = \{(1,3),(2,2),(3,1)\}$. Note that

$$\sum_{k=1}^{3} \left(\frac{(-1)^k}{k!} \sum_{(a_1,\cdots,a_k) \in S_k(3)} \frac{1}{a_1 \cdots a_k} \right) = -\frac{1}{1} \cdot \frac{1}{3} + \frac{1}{2!} \left(\frac{1}{1.2} + \frac{1}{2.1} \right) - \frac{1}{3!} \cdot \frac{1}{1.1.1} = 0.$$

Prove that for all $n > 1$ that

$$\sum_{k=1}^{n} \left(\frac{(-1)^k}{k!} \sum_{(a_1,\cdots,a_k) \in S_k(n)} \frac{1}{a_1 a_2 \cdots a_k} \right) = 0.$$

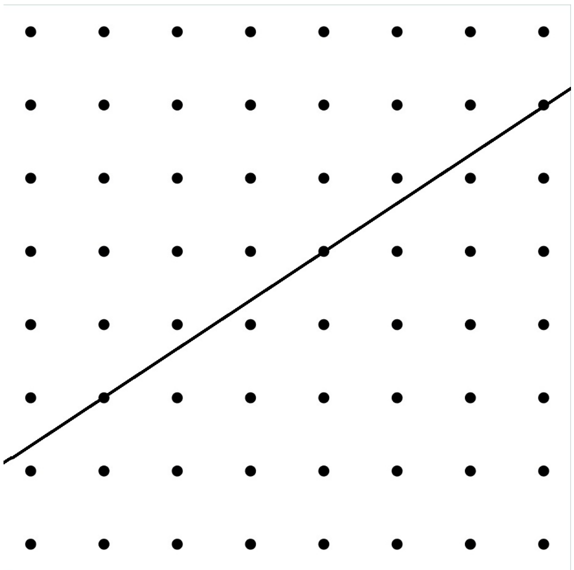
Solution for the unsolved problem 5 from MS 94(1-2) 2025.

MS 94(1-2) 2025 : Problem 5 (by **Dr. Chudamani Pranesachar Anil Kumar**, KREA University, India).

Let

$$GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\}$$

be the group of (2×2) integer matrices of determinant ± 1. Consider the integer grid $\mathbb{Z} \times \mathbb{Z}$ as shown in the figure below. Let $\mathcal{L}$ be the set of lines obtained in the plane by extending the line segment joining any two points of the integer grid. Define an action of $GL_2(\mathbb{Z})$ on the set $\mathcal{L}$ as



follows. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$. Let $L \in \mathcal{L}$ be a line joining (say) two points $P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, P_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{Z} \times \mathbb{Z}$ of the integer grid. Then $T \cdot L$ is defined as the line joining two points

$Q_1 = T \cdot P_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{pmatrix}$ and

$Q_2 = T \cdot P_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} ax_2 + by_2 \\ cx_2 + dy_2 \end{pmatrix}$.

The action of $GL_2(\mathbb{Z})$ on the set $\mathcal{L}$ of lines decomposes $\mathcal{L}$ into disjoint union of orbits.

Prove the following:

(1) Let $L \in \mathcal{L}$ and suppose L is the line joining $P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, P_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{Z} \times \mathbb{Z}$ of the integer grid.

Show that there exists an integer grid point $P = \begin{pmatrix} x \\ y \end{pmatrix}$ on the line L such that $gcd(x, y) = 1$.

(2) All lines in $\mathcal{L}$ passing through the origin lie in one single orbit for the $GL_2(\mathbb{Z})$-action.

(3) There are infinitely many distinct orbits in $\mathcal{L}$ for the $GL_2(\mathbb{Z})$ action.

(4) Describe all the orbits.

Solution: (by the Proposer)

We prove (1). Let $L \in \mathcal{L}$ be a line passing through the origin. Since there exists one more lattice point on the line L other than the origin, let P be the nearest nonzero lattice point to the origin on the line L. Let $P = (x_0, y_0)^t$. We show that $gcd(x_0, y_0) = 1$. The equation of the line L is given by $x_0 y = y_0 x$ since it contains the origin and the point P. If $gcd(x_0, y_0) = d > 0$ then we have that $Q = (\frac{x_0}{d}, \frac{y_0}{d})^t$ also lies on the line L because it satisfies the equation of the line L. Since distance from the origin to Q is less than or equal to the distance from the origin to P and P is the nearest nonzero point on the line L, we must have $P = Q$ and $d = 1$. Hence in this case we have found a point $P = (x_0, y_0)^t$ on the line L such that $gcd(x_0, y_0) = 1$.

Let $L \in \mathcal{L}$ be a line not passing through the origin. Since there are two lattice points on the line L, let $(a, b)^t$ and $(c, d)^t$ be two consecutive lattice points on the line L. Then by translating the line L to the line $L' = \{(x - a, y - b)^t \mid (x, y)^t \in L\} = L - (a, b)^t$, we find that $L' \in \mathcal{L}$ passes through the origin and the point $(c - a, d - b)^t$ is the nearest nonzero lattice point on the line L', nearest to the origin. Hence by the previous argumentin the last paragraph, we must have $gcd(c - a, d - b) = 1$. So there exist integers p, q such that $p(c - a) + q(d - b) = 1$.

Consider the point $R = \big(X = a + t(c - a), Y = b + t(d - b)\big)^t$ on the line L where $t = 1 - pa - qb$. Note that the point R also lies on the same line L which passes through the lattice points $(a, b)^t, (c, d)^t$. Now we have

$$\begin{aligned} pX + qY &= pa + pt(c - a) + qb + qt(d - b) \\ &= pa + qb + t\big(p(c - a) + q(c - b)\big) \\ &= pa + qb + t = 1. \end{aligned}$$

Hence $gcd(X, Y) = 1$ and we have proved in this case also, when the line L is not passing through the origin, that there exists a point $(x, y)^t$ on L such that $gcd(x, y) = 1$.

We prove (2). Let L_1 and L_2 be two lines passing through the origin and let $(x_1, y_1)^t \in L_1, (x_2, y_2)^t \in L_2$ such that $gcd(x_1, y_1) = 1 = gcd(x_2, y_2)$. Therefore there exist integers u_1, v_1, u_2, v_2 such that

$u_1x_1 + v_1y_1 = 1 = u_2x_2 + v_2y_2 = 1$. Hence Consider the matrix

$$M_1 = \begin{pmatrix} u_1 & v_1 \\ -y_1 & x_1 \end{pmatrix}, M_2 = \begin{pmatrix} u_2 & v_2 \\ -y_2 & x_2 \end{pmatrix}.$$

We have $\text{Det}(M_1) = 1 = \text{Det}(M_2) \Rightarrow M_1, M_2 \in GL_2(\mathbb{Z})$ and we also have

$$M_1 \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = M_2 \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Hence we take

$$M = M_2^{-1} \cdot M_1 = \begin{pmatrix} x_2 & -v_2 \\ y_2 & u_2 \end{pmatrix} \cdot \begin{pmatrix} u_1 & v_1 \\ -y_1 & x_1 \end{pmatrix} \in GL_2(\mathbb{Z}).$$

Now we observe that M takes origin to the origin and

$$M \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Hence the matrix $M \in GL_2(\mathbb{Z})$ takes the line L_1 to the line L_2.
This proves that all lines passing through the origin forms a single orbit.

We prove (3). Consider two consecutive points $P = (n, 1)^t, Q = (n, 0)^t$ on the line L_n. Now for any matrix $M \in GL_2(\mathbb{Z})$ we must have that the points

$$M \cdot \begin{pmatrix} n \\ 1 \end{pmatrix}, M \cdot \begin{pmatrix} n \\ 0 \end{pmatrix}$$

are two consecutive points on the line T which is the image of L_n under the map M.
Now for any two consecutive points $(n, k+1)^t, (n, k)^t$ on the line L_n we have that the matrix

$$\begin{pmatrix} n & n \\ k+1 & k \end{pmatrix}$$

has determinant equal to n in absolute value.
If $T = L_m$ for some $m > 0$ then the matrix containing two consecutive points

$$M \cdot \begin{pmatrix} n \\ 1 \end{pmatrix}, M \cdot \begin{pmatrix} n \\ 0 \end{pmatrix}$$

must have determinant equal to m in absolute value. But we see that

$$\left(M \cdot \begin{pmatrix} n \\ 1 \end{pmatrix} \; M \cdot \begin{pmatrix} n \\ 0 \end{pmatrix} \right) = M. \begin{pmatrix} n & 1 \\ n & 0 \end{pmatrix}$$

and hence it has absolute value of the determinant equal

$$| \operatorname{Det}(M) \operatorname{Det}(\begin{pmatrix} n & 1 \\ n & 0 \end{pmatrix}) |= n.$$

Therefore we must have $n = m$. Hence we conclude that the lines L_n for $n \geq 1$ lie in distinct orbits and there are infinitely many orbits.

We prove (4). Let us prove that in each orbit there exists line $L_n = \{(n, y)^t \mid y \in \mathbb{R}\}$ for a unique $n \in \{0, 1, 2, 3, \cdots, \}$.
Consider a line L and take two consecutive points $(a, b)^t, (c, d)^t$ on the line L and such that $gcd(a, b) = 1$. Such a point $(a, b)^t \in L$ with $gcd(a, b) = 1$ always exists as we have seen before. Let the absolute value of the determinant of the matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

be $n \in \{0, 1, 2, 3, \cdots\}$. Note that the absolute value of the determinant of the matrix containing two consecutive points of a line does not change for any two consecutive points of the same line. Now we prove that the orbit O, containing L, contains the line L_n. Without loss of generality by using the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and the line $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot L$ we can assume that $ad - bc = n > 0$. The case $n = 0$ is straightforward and we have already done this case.
There exist integers p, q such that $pa + qb = 1$. Let $M = \begin{pmatrix} p & q \\ -b & a \end{pmatrix}$. Then we have

$$M \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & e \\ 0 & n \end{pmatrix} = N$$

for some integer e. Since the points $(1, 0)^t$ and $(e, n)^t$ are consecutive on the line $M \cdot L$, the difference point $(e, n)^t - (1, 0)^t = (e - 1, n)^t$ has the property that $gcd(e - 1, n) = 1$. So there exist integers k, l such that $(e - 1)k + ln = 1$. Consider the matrix

$$R = \begin{pmatrix} n & n \\ k & k+1 \end{pmatrix}.$$

We have

$$S = R \cdot N^{-1} = \frac{1}{n} \begin{pmatrix} n & n \\ k & k+1 \end{pmatrix} \cdot \begin{pmatrix} n & -e \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n & -e+1 \\ k & \frac{(-e+1)k+1}{n} = l \end{pmatrix} \in GL_2(\mathbb{Z}).$$

Now we observe that

$$S \cdot \begin{pmatrix} 1 & e \\ 0 & n \end{pmatrix} = \begin{pmatrix} n & n \\ k & k+1 \end{pmatrix}.$$

Therefore we have

$$S \cdot M \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} n & n \\ k & k+1 \end{pmatrix}.$$

Now the two points $(n,k)^t, (n,k+1)^t$ are consecutive points on the line L_n and the matrix $S \cdot M \in GL_2(\mathbb{Z})$. Hence we have the mapped the line L to the vertical line L_n by using a matrix in $GL_2(\mathbb{Z})$. Here "n" is nothing but the absolute value of the determinant of the matrix containing two consecutive points of L.
We have characterized all the orbits by the absolute value of this determinant value. Two lines in $\mathcal{L}$ are in the same orbit of $GL_2(\mathbb{Z})$ if and only if the absolute value of the determinant of the matrices containing two consecutive points of the two lines are equal.
This completes the proof of the problem.

Solutions for the problems 1 and 4 from MS 94(3-4) 2025.

MS 94(3-4) 2025 : Problem 1 (by **Dr. Quang Hung Tran**, High School for Gifted Students, Vietnam University of Science, Vietnam).

Consider a skew polygon $A_1A_2\ldots A_n$, in the Euclidean space $\mathbb{E}^d$, $n, d \in \mathbb{N} \setminus \{1\}$. Prove that for any point $P \in \mathbb{E}^d$, there always exist $\alpha_i \in \{-1, 0, 1\},\ i = 1,\, 2,\, \ldots\, n$ such that

$$\left|\sum_{j=1}^{n} \alpha_j \mathbf{PA_j}\right|^2 \geq \frac{1}{4}\left((A_1A_2)^2 + (A_2A_3)^2 + \cdots + (A_{n-1}A_n)^2 + (A_nA_1)^2\right),$$

where $\mathbf{PA_j}$ is the vector from P to A_j, $j = 1,\, 2,\, \cdots, n$.

This problem was solved by Mr. Santanu Panda, Hooghly, West Bengal, India.

Solution: (by the Proposer)

We need the following lemma:

Given n $(n \geq 2)$ vectors $\vec{u_1},\, \vec{u_2}\ldots,\, \vec{u_n}$ Euclidean space $\mathbb{E}^d$ $(d \geq 2)$. Prove that it is always possible to choose addition or subtraction signs such that

$$|\pm\vec{u_1} \pm \vec{u_2} \pm \ldots \pm \vec{u_n}|^2 \geq |\vec{u_1}|^2 + |\vec{u_2}|^2 + \ldots + |\vec{u_n}|^2.$$

Proof. We proceed by induction on n.

For $n = 2$, it is always possible to choose $\pm$ signs such that $|\vec{u_1} \pm \vec{u_2}|^2 \geq |\vec{u_1}|^2 + |\vec{u_2}|^2$ because if $(\vec{u_1}, \vec{u_2}) \geq 90^\circ$, choose $-$; otherwise, choose $+$.

Assume the statement is true for $n = k$, i.e., it is always possible to choose $\pm$ signs such that

$$|\pm\vec{u_1} \pm \vec{u_2} \pm \ldots \pm \vec{u_k}|^2 \geq |\vec{u_1}|^2 + |\vec{u_2}|^2 + \ldots + |\vec{u_k}|^2 .$$

We shall prove the statement for $n = k+1$. For $n = k+1$, let

$$\vec{v} = \pm\vec{u_1} \pm \vec{u_2} \pm \ldots \pm \vec{u_k}$$

then, by the inductive hypothesis, it is always possible to choose $\pm$ signs such that

$$|\vec{v}|^2 \geq |\vec{u_1}|^2 + |\vec{u_2}|^2 + \ldots + |\vec{u_k}|^2 .$$

Moreover, with two vectors $\vec{v}$ and $\vec{u_{k+1}}$, it is also possible to choose $\pm$ signs such that

$$|\vec{v}^2 \pm \vec{u_{k+1}}|^2 \geq |\vec{v}|^2 + |\vec{u_{k+1}}|^2.$$

Combining these two inequalities, we have proven the statement for $n = k+1$. □

Back to the main problem. Let $\vec{u_1} = \vec{A_1A_2}, \ldots, \vec{u_n} = \vec{A_nA_1}$. Applying the lemma, we can always choose $\pm$ signs such that

$$\left|\pm\vec{A_1A_2} \pm \vec{A_2A_3} \pm \ldots \pm \vec{A_nA_1}\right|^2 \geq A_1A_2^2 + A_2A_3^2 + \ldots + A_nA_1^2.$$

For any point P, we deduce that we can always choose $\pm$ signs such that

$$\left|\pm\left(\vec{PA_2} - \vec{PA_1}\right) \pm \left(\vec{PA_3} - \vec{PA_2}\right) \pm \ldots \pm \left(\vec{PA_1} - \vec{PA_n}\right)\right|^2$$
$$\geq A_1A_2^2 + A_2A_3^2 + \ldots + A_nA_1^2.$$

This implies that we can always choose $\pm$ signs such that

$$\left|\pm\vec{PA_1} \pm \vec{PA_1} + \pm\vec{PA_2} \pm \vec{PA_2} \ldots \pm \vec{PA_n} \pm \vec{PA_n}\right|^2$$
$$\geq A_1A_2^2 + A_2A_3^2 + \ldots + A_nA_1^2.$$

Note that the coefficients of $\vec{PA_1}$ in the above inequality always belong to $\{-2, 0, 2\}$. Therefore, dividing both sides of the inequality by 4, we obtain that we can always choose $\alpha_i \in \{-1, 0, 1\},\ i = 1 \ldots n$ such that

$$\left|\alpha_1\vec{PA_1} + \alpha_2\vec{PA_2} + \ldots + \alpha_n\vec{PA_n}\right|^2 \geq \frac{1}{4}\left(A_1A_2^2 + A_2A_3^2 + \ldots + A_{n-1}A_n^2 + A_nA_1^2\right).$$

There is equality when $n = 4$ and $A_1A_2A_3A_4$ is a square, with $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 0, \alpha_4 = 0$. This completes the proof. □

MS 94(3-4) 2025 : Problem 4 (by **Dr. B. Sury**, ISI, Bengaluru, India).

Given positive integers m, n, prove that there are infinitely many positive integers N such that the binomial coefficient $\binom{N}{m}$ is relatively prime to n.

This problem was solved independently by Mr. Santanu Panda, Hooghly, West Bengal, India and Ms. Manika Gupta, Institute of Mathematical Sciences, Chennai, India.

Solution: (by Mr. Santanu Panda)

For any positive integer k , let $N = knm!+m$. Then, there exists a positive integer ϕ such that

$$\binom{N}{m} = \frac{(knm! + m)(knm! + m - 1) \cdots (knm! + 1)}{m!} = \frac{\phi knm! + m!}{m!} = \phi kn+1.$$

So $\binom{N}{m} = \phi kn + 1$, which implies that $\binom{N}{m}$ is relatively prime to n . As k takes infinitely many values, there are infinitely many positive integers N such that the binomial coefficient $\binom{N}{m}$ is relatively prime to n.

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