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THE MATHEMATICS STUDENT

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(Issued: April, 2023)

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M. M. SHIKARE

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Edited by M. M. SHIKARE

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PROFESSOR SATYA DEO TRIPATHI: A MATHEMATICIAN AND A GOOD HUMAN BEING

N. K. THAKARE

Professor Satya Deo felt uneasiness while he was attending an international conference in New Delhi on 18th June 2022. He was rushed to hospital; but human efforts seemed inadequate to save his life. A Mathematician died on 19th June 2022 in Mathematical atmosphere.

Professor Satya Deo breathed his last as General Secretary of the Indian Mathematical Society (IMS), the oldest scientific society in the country. He died while he was also a General Secretary of the National Academy of Sciences, India (NASI), Allahabad. His passing away as left voids in both the Indian Mathematical Society and in NASI.

I had a specific soft corner for Professor Satya Deo on account of things that are rather personal.

He was an alumnus of Arkansas State University, Fayetteville, USA, the place very close to the Head Quarters of Walmart (one of the largest companies in the world) where my son is one of the top officers of Walmart. I consider myself lucky to have proposed his Fellowship Nomination that was seconded by Late Professor H. C. Khare; and he became the Fellow of the National Academy of Sciences, India.

Our relations were so cordial that when he organized 65th Annual Conference of the Indian Mathematical Society at Rewa (Madhya Pradesh), he asked me to visit Rewa and help his colleagues to prepare for the said conference as he was very busy being the Vice Chancellor of Awadhesh Pratap Singh University, Rewa. I stayed there for a few days and during that period

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I helped his colleagues to complete formalities including seeking funds for financial support from NBHM, DST, CSIR, UGC and such funding agencies.

He was also the Vice Chancellor of Rani Durgawati University, Jabalpur (Madhya Pradesh). Earlier he was on the faculty of Rani Durgawati University and organized several refresher courses in Mathematics as well as conferences in Mathematics. He was so warm to me that he asked me to deliver lectures during those events on diverse topics such as i) Combinatorics on finite sets ii) Geometry of Banach Spaces, iii) Biorthogonal Polynomials and popular talks such as: a) Mathematics is an Art, b) Mathematicians as Human Beings etc. In fact, on few occasions local newspapers carried out reports in praise of Satya Deo for organizing such popular lectures which could be understood by a layman.

He was the President of the Indian Mathematical Society and he Presided over the deliberations of the 66th Annual Conference of IMS held at Aurangabad (Maharashtra) in December 2000. Prof Satya Deo was a well-known algebraic topologist. He was well respected for his quality researches. After his return to India from USA he worked in universities such as Delhi, Jammu, Jabalpur etc. and ultimately, he settled at the Harish-Chandra Research Institute at Allahabad. He was very happy with his association with HRI.

During the centenary year of existence of the Indian Mathematical Society (i.e. 2007), Prof. Satya Deo was the Academic Secretary of the Indian Mathematical Society. Qualitatively the academic program that he arranged was of international standard. That conference was graced by Abel Laureate S. R. S. Varadhan, Fields Medalist S. T. Yau, Clay Prize Winner Prof. Richard Hamilton, Prof. S. S. Abhyankar, Prof. M. Ram Murty and such eminent and talented Mathematicians. Prior to his taking over as an Academic Secretary of the IMS, I had the good fortune to be an Academic Secretary of IMS for nine long years. Because of the warm relations between us he involved me to the best of my abilities in preparing academic program during that year and subsequently too.

He was the Academic Secretary of IMS for six years. In 2013 he took over from me the editorship of the Journal of IMS. He remained editor for six years and helped Journal of IMS to grow. He became General Secretary of IMS from 1st April 2019 as I decided not to hold any office of IMS. During his tenure as General Secretary, many good things happened such as acquiring a land admeasuring more than an acre at Pune for construction of permanent headquarters of IMS. In spite of the pandemic of Corona-19 virus, he achieved a great deal of succuss as the General Secretary of IMS.

Prof Satya Deo was a man with robust physique with a height of almost 6 feet. He was a man with strong mind and kind heart. Unfortunately, he suffered from a failure of a kidney and for last many years he was required to go for dialysis twice per week (i. e. on Thursdays and Sundays every week). For last two three years the frequency of dialysis was increased to three days per week as Tuesday became the additional dialysis day for him.

Prof Satya Deo valued ethics very much. An incident which took place in that context is worth reporting. In the Indian Mathematical Society some conventions and traditions are established and the office bearers try to adhere to them. On the opening day of the Annual Conference of the Indian Mathematical Society the President of IMS delivers his Presidential Address (technical) immediately after the inauguration of the conference. One such convention is that Sr. most past president of IMS who is present for the conference, presides over the said technical address. For many years that mantle fell upon me. However, in one instance a very senior past president was attending the conference and as such being the General Secretary of the IMS, I was on the verge of proposing that senior most past president to preside over the presidential address for that conference. However, Prof Satya Deo pulled me to a corner and insisted upon me not to request the said Mathematician to preside over that session. “Why?” I asked to him. Prof Satya Deo explained that the said Mathematician has been convicted and hence it will be inappropriate to offer him the dignity of presiding over a prestigious address. He was so insistent that I had to give in and I myself was constrained to preside over the said presidential address. It resulted in breaking our long-standing convention. But what was important is Prof

Satya Deo's ethical attitude.

We both hailed from village background. Prof Satya Deo from UP village and myself from Maharashtra village. Once he told me that his close relatives still rear cattle in his ancestral house besides cultivating their agricultural farms. After retirement I also tried to do farming. But I failed miserably. I also tried to rear cattle, but there also I failed miserably.

I offer my sincere and deep condolences to the bereaved family of Prof Satya Deo. We shall always remember Prof Satya Deo for enduring work he did for the Indian Mathematical Society.

We shall never forget you Prof. Satya Deo!

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PROFESSOR SATYA DEO AS I KNEW HIM

SUKUMAR DAS ADHIKARI

My early acquaintance with Professor Satya Deo was in some conferences in Topology, but I came to know him better when I visited him in the Department of Mathematics and Computer Science of Rani Durgawati Vishwavidyalaya, Jabalpur, during March 24-26, 2003. Prof. T. Pati, an ex vice chancellor of Allahabad University, was also visiting Prof. Satya Deo at that time and I could listen to many interesting stories during their conversation at a lunch hosted by Prof. Satya Deo.

Soon after that, in July 2003, Prof. Satya Deo joined Harish-Chandra Research Institute (HRI), Allahabad, as a visiting professor. At HRI, I had opportunities to interact with him on a personal level. He used to take classes for the students at HRI and used to lecture in various training programmes held at HRI for several years and was known as a great teacher. At some point, he was lecturing to me twice a week on some results in topological methods in combinatorics. We had a plan to work on some problems together; but unfortunately this did not materialize, due to my moving to Kolkata.

He had worked in several Universities; in all these places he had many admirers, both for his dedication to his work and his charming personality. Despite having severe health problems for many years, he continued to work hard, specially for the Indian Mathematical Society and the National Academy of Sciences, India. He also continued to publish papers and help students in research till the very end.

Personally he was like my elder brother, ready to extend help in any difficult situation. Several times he helped me when I had some health problem. Like many others, I used to value his advise in difficult times. In

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spite of his very busy schedule, he would make time to meet and talk to a group of us regularly around 11 am on tea at HRI pantry. There was a time when his doctor advised him a strict fluid restriction; in those days he stopped taking tea but would come to meet us in the pantry at 11 am.

I am close to his eldest son, Dr. Satya Prakash Tripathi, who is also a Mathematics Professor in Delhi; from him I came to know many things related to his father. Let me record some of them here.

Early in the life, Prof Satya Deo, due to poor financial condition of his family, had applied for a job in railways after his B.Sc. and got selected. But the official appointment letter did not come for more than a year. In the meantime, he joined M.Sc. Mathematics at Allahabad University and got very interested in pursuing mathematics as a career. He declined the job in the railways and, after finishing M.Sc. as a topper, he got the offer to become a lecturer at Ewing Christian College, Allahabad University, Allahabad, in 1966, which he gladly accepted. This was a big turning point in his life and career. Later, after obtaining his Ph.D. at the University of Arkansas, Fayetteville, in Algebraic Topology, he taught in several Universities and also shouldered some of the highest administrative responsibilities (he was Vice Chancellor of two universities in Madhya Pradesh viz., APS University, Rewa and R. D. University, Jabalpur).

Outside Mathematics, he used to love listening to good music and playing chess. When he was young, he was very good at playing flute, though he stopped playing flute in the later years. However, he could also perform very well on drums and he never missed his performance on drums during Holi festival, and his friends and relatives loved to see him playing drums. In the family life, though he was very kind and caring, he did not like pampering his children lest they forget to value time and money and used to advise them to do the hard work and stand on their own feet.

We knew that he would be going to Delhi to receive the Life Time Achievement award in the field of Mathematics from Vijnana Parishad of India in June 2022; but the sad news came that he had collapsed during the award function due to a massive heart attack just after receiving the award and had passed away on June 19, 2022. The sudden demise of Prof Satya Deo, deeply shocked and saddened the mathematics community. It is an irreparable loss for the Indian Mathematical Society. His memory will continue to inspire us.

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DISCRETE STORIES : STRAY FACTS RELATED TO SOME EARLY RAMSEY-TYPE THEOREMS*

SUKUMAR DAS ADHIKARI

I would like to thank the members of the Indian Mathematical Society for electing me the President of the Society for the year 2022-2023. It is a great honor and privilege to address the mathematicians and the guests attending the conference.

I looked into some of the articles in Math. Student based on the past Presidential Addresses (General) in Annual Conferences of the IMS. One sees valuable articles: emphasizing the role of history in learning and teaching mathematics; informing us about mathematicians, institutes devoted to mathematics and conferences on mathematics; suggestions to the policy makers and advises to teachers and students; reminding us about our mathematical heritage; explaining the role of various prizes and other recognitions towards the promotion of mathematics; and so on.

Discussion on mathematics teaching at the school level would be a topic of interest; however even if one has been involved in training students at that level, it requires much more research before one embarks on talking on this theme. Though some institutes and individuals are doing their best through many training programmes etc., to avoid looking into school teaching in general, will be like looking for food supplements as a substitute for whole foods. It is a rather complex topic which involves several socio-economic issues. So I decided to eschew this issue in today's address, and instead took up a theme for my address, issuing from a point raised in the article by Prof. B. Sury, "View of mathematics by our society and what our role could be"; this incidentally was based on his talk at the 86th Annual

* This article is based on the general talk given by Prof. S. D. Adhikari, the president of IMS, in the 88th Annual Conference of the IMS-An International Meet held at BIT, Mesra, Ranchi, Jharkhand during December 27-30, 2022.

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Conference of the IMS, and it drew my attention to the need for popular writings to create general awareness about mathematics.

I decided to take up the present theme, which will require only high school mathematics and can be explained by simple examples drawn from 'common sense' situations as Prof. Sury would suggest. At the same time, the origin of many important recent developments in mathematics can be traced back to these results.

The classical Ramsey theorem (1930) can be said to be a generalization of the pigeonhole principle. The pigeonhole principle says that if $kn + 1$ objects are put in n pigeonholes, then there will be a pigeonhole containing at least $k + 1$ objects; it is a consequence of a simple counting argument.

As a first case of Ramsey's theorem, one has the following well-known high school exercise:

If you have a party of at least 6 people, you can guarantee that there will be a group of 3 people who all know each other, or a group of 3 people who all do not know each other. In the language of graph theory it says that, if you have a complete graph with six vertices and colour the edges with two colours, say red and blue, then there will be a monochromatic triangle.

The above statement follows easily from the pigeonhole principle, stated above.

Ramsey's theorem, a generalization of the party problem above, appeared as a lemma in a paper (1930) of Frank Plumpton Ramsey on Mathematical logic. Ramsey passed away in January 1930 at the age of 26. It should be mentioned that Ramsey was mainly interested in philosophy and in spite of his passing away at a very young age, he made remarkable contributions to mathematical economics.

Three years after the publication of Ramsey's theorem, a different proof was given by Skolem. While Skolem was aware of the result, Paul Erdős and George Szekeres were led independently to this result while solving a problem brought to them by their friend Esther Klein; Paul, George and Esther were about 19, 21 and 22 years old respectively at that time. Soon afterwards, Erdős and Szekeres ran into the paper of Ramsey. The problem of Esther Klein, which was solved by Erdős and Szekeres was about ascertaining the existence of an integer $ES(n)$, such that any $ES(n)$ points in the plane in general position, will have a subset of n points forming a convex polygon. Erdős named the theorem the "happy ending problem" as it led to the marriage of Esther Klein and George Szekeres. As Soifer has mentioned in his book ‘Ramsey Theory: Yesterday, Today, and Tomorrow", after enjoying a life full of mathematics and cheer, George Szekeres and Esther Klein Szekeres passed away on the same day (on August 28, 2005). Paul Erdős, who was slightly younger among the three, had already passed away in 1996.

Subsequently, the branch of combinatorics called Ramsey Theory grew in stature and significance and with hindsight we can now see the unifying feature of the early Ramsey-type theorems which are seemingly unrelated. Some results due to Schur, van der Waerden and Hilbert, share the credit of preceding the result of Ramsey in the class of Ramsey-type theorems.

Commenting on Ramsey's being called as ‘eponymous’ by Mellor (1983), Harary in his tribute (1983) to Frank P. Ramsey says:

“ Of course! The study of ramsey theory has become so important that his name has become an adjective, along with other immortal mathematical lower case adjectives, including abelian, boolean, cartesian, ...”

Existence of regular substructures in general combinatorial structures is the phenomena which can be said to characterize the subject of Ramsey theory. Most often, we come across results saying that:

“If a large structure is divided into finitely many parts, at least one of the parts will retain certain regularity properties of the original structure.”

In some results in Ramsey theory, ‘Large’ substructures are seen to have certain regularities.

The statement of Theodore Motzkin that “Complete disorder is impossible" is perhaps the best to describe the philosophy of Ramsey Theory in a nutshell.

Instead of taking up the developments involving the generalizations of the early Ramsey-type theorems (due to Schur, van der Waerden and Hilbert), I will confine myself to making some remarks on the theorem of van der Waerden.

The theorem of van der Waerden is one among the 'pearls' that Khinchin presented in his 'Three pearls of Number Theory'; apart from van der Waerden's theorem, this small book also talks about the Landau-Schnirelman hypothesis and Waring's problem. Incidentally, this book was written by Khinchin for a soldier (Seryozha) in the second world war, who, lying in a hospital, had written to Khinchin for something mathematical to study and pass his time.

The theorem of van der Waerden has led to many interesting developments in Combinatorics and Number Theory. It says:

Given positive integers k and r, there exists a positive integer $W(k,r)$ such that for any r-colouring of $\{1,2,\ldots W(k,r)\}$, there is a monochromatic arithmetic progression of k terms.

Regarding van der Waerden's theorem, we are lucky to have van der Waerden's personal account (1971): 'How the proof of Baudet's conjecture was found". It contains the formulation of the problem with the valuable suggestions due to Emil Artin and Otto Schreier and depicts how the sequence of basic ideas occurred as an elaboration of the psychology of invention. It seems the result was independently conjectured by Schur and Baudet; since van der Waerden came to know it through Baudet, he calls it Baudet's conjecture.

I add some comments before I close. A theorem due to Hales and Jewett (1963) revealed the combinatorial nature of van der Waerden's theorem, showing that this 'pearl of number theory' belongs to the ancient shore of Combinatorics. Today we see that van der Waerden's theorem was a prelude to a very important theme where interplay of several areas of Mathematics would be seen. One had the results of Roth and Szemerédi and a number of different proofs of these results including the ergodic proof of Szemeredi's theorem due to Furstenberg, which started the subject of Ergodic Ramsey Theory. Then came the results of Gowers and the Green-Tao Theorem. Many challenging open questions are still remaining to be answered.

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SOME ZERO-SUM CONSTANTS AND THEIR WEIGHTED GENERALIZATIONS*

SUKUMAR DAS ADHIKARI

Abstract. “Zero-sum problems form another developing area in Additive Combinatorics having several applications. The Erdős -Ginzburg-Ziv Theorem was the starting point of this area", was the remark made by Y. O. Hamidoune and I. P. da Silva. Here, after describing some results and open questions related to the classical zero-sum constants, we talk about some developments around their weighted generalizations.

1. Introduction

Investigations into zero-sum problems were initiated by a result of Erdős, Ginzburg and Ziv (known as *the EGZ-theorem*) [33], published in 1961.

If G is a finite abelian group (written additively) and $S = (g_1, g_2, \dots, g_l)$ a sequence of elements of G, not necessarily distinct, S is called a *zero-sum sequence* if

$$g_1 + \cdots + g_l = 0,$$

where 0 is the identity element of the group.

We now state the EGZ-theorem.

Theorem 1.1. *Every sequence of length $2n-1$ of elements in an abelian group of order n contains a zero-sum subsequence of length n.*

Sketch of a proof. It is not difficult to see that the essence of the EGZ theorem lies in the case when $G = \mathbb{Z}_p$, p is a prime.

One invokes the following generalized version of the Cauchy-Davenport inequality [26], [28] (see also [29]):

* This article is based on the Technical talk given by Prof. S. D. Adhikari, the president of the IMS, in the 88th Annual Conference of the IMS-An International Meet held at BIT, Mesra, Ranchi, Jharkhand during December 27-30, 2022.

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Let $A_1, A_2, \cdots, A_h$ be non-empty subsets of $\mathbb{Z}_p$. Then for a finite set X, denoting its cardinality by $|X|$,

$$|\sum_{j=1}^{h} A_i| \geq \min\left(p, \sum_{i=1}^{h} |A_i| - h + 1\right).$$

Considering representatives modulo p in the interval $0 \leq a_i \leq p-1$ for the given elements and rearranging, if necessary, we have

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_{2p-1} \leq p-1.$$

We can now assume that

$$a_j \neq a_{j+p-1}, \text{ for } j = 1, \cdots, p-1.$$

Now, writing $A_i := \{a_j, a_{j+p-1}\}$, for $j = 1, \cdots, p-1$,
by the Cauchy-Davenport inequality, we have

$$|\sum_{j=1}^{p-1} A_i| \geq \min\left(p, \sum_{i=1}^{p-1} |A_i| - (p-1) + 1\right) = p,$$

so that

$$-a_{2p-1} \in \sum_{j=1}^{p-1} A_i.$$

□

There are many proofs of the above theorem available in the literature (see [1], [16], [18], [54] for instance).

Soon after the EGZ theorem, during 1963-66, Rogers [59], Davenport (see [55]) and some others posed the problem of determining the smallest natural number k such that any sequence of k elements in G has a non-empty zero-sum subsequence; it is known as *the Davenport constant* in the literature and is denoted by $D(G)$. Defined originally in connection with non-unique factorization in algebraic number theory, it had applications in graph theory (see, for instance, [25] or [37]) and in the proof of the infinitude of Carmichael numbers by Alford, Granville and Pomerance [19]. It was also useful in an interesting paper on Number Theory written by Brüdern and Godinho [23].

We try to have a rough idea of what is known about $D(G)$.

$$\text{Given } G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_r},$$

with $n_1|n_2|\cdots|n_r$, writing

$$M(G) = 1 + \sum_{i=1}^{r}(n_i - 1), \tag{1.1}$$

it is trivial to see that

$$M(G) \leq D(G) \leq |G|.$$

The equality $D(G) = |G|$ holds if and only if $G = \mathbb{Z}_n$, the cyclic group of order n.

Olson [55] [56] proved that $D(G) = M(G)$ for all finite abelian groups of rank 2 and for all p-groups. It is also known that $D(G) > M(G)$ for infinitely many finite abelian groups of rank $d > 3$ (see [39], for instance). Some interesting results on the upper bound of $D(G)$ for a non-cyclic abelian group G, can be found in the papers of Caro [24], Ordaz and Quiroz [57] and Balasubramanian and Bhowmik [21].

Emde Boas and Kruyswijk [22], Baker and Schmidt [20] and Meshulam [52] gave upper bounds involving the exponent of the group and the cardinality of the group G and the best among them (proved again by Alford, Granville and Pomerance [19]) is:

$$D(G) \leq m\left(1 + \log\frac{|G|}{m}\right),$$

where m is the exponent of G.

We have the following conjectures.

(1) **(Gao and Geroldinger,** [35] **and** [36]**)**
$D(G) = M(G)$, for all G with rank $d = 3$ or $G = \mathbb{Z}_n^d$.

(2) **(Narkiewicz and Śliwa,** [53]**)** $D(G) \leq \sum_{i=1}^{d} n_i$.

For a finite abelian group G, the definitions of the following two constants were suggested by the EGZ-theorem:

- $E(G)$ is the smallest natural number k such that any sequence of k elements in G has a zero-sum subsequence of length $|G|$.
- $\mathsf{s}(G)$ is the smallest natural number k such that any sequence of k elements in G has a zero-sum subsequence of length $\exp(G)$.

The following beautiful result of Gao [34] (see also [38], Proposition 5.7.9) connects the constants $D(G)$ and $E(G)$.

Theorem 1.2. (Gao) *For a finite abelian group G, we have*

$$E(G) = D(G) + |G| - 1. \tag{1.2}$$

The higher dimensional analogue of the EGZ- theorem, which was considered initially by Harborth [48] and Kemnitz [49], is the study of the constant $\mathsf{s}(\mathbb{Z}_n^d)$.

Harborth observed the following trivial bounds:

$$1 + 2^d(n-1) \leq \mathsf{s}(\mathbb{Z}_n^d) \leq 1 + n^d(n-1). \tag{1.3}$$

Another observation made by Harborth is the following:

$$\mathsf{s}(\mathbb{Z}_{mn}^d) \leq \min\Big(\mathsf{s}(\mathbb{Z}_n^d) + n(\mathsf{s}(\mathbb{Z}_m^d) - 1), \mathsf{s}(\mathbb{Z}_m^d) + m(\mathsf{s}(\mathbb{Z}_n^d) - 1)\Big). \tag{1.4}$$

Equations (1.3) and (1.4) give the exact value for $\mathsf{s}(\mathbb{Z}_{2^e}^d)$ for any $d \geq 2$.

For general n, the EGZ theorem gives the exact value

$$\mathsf{s}(\mathbb{Z}_n) = 2n - 1.$$

For the case $d = 2$ also, the lower bound in (1.3) was expected to give the right magnitude of $\mathsf{s}(\mathbb{Z}_n^2)$ and this expectation, had been known as *the Kemnitz Conjecture* in the literature. Alon and Dubiner [16] had established that $\mathsf{s}(\mathbb{Z}_n^2) \leq 6n - 5$ and Rónyai [60] had proved that for a prime p, one has $\mathsf{s}(\mathbb{Z}_p^2) \leq 4p - 2$, which implies that $\mathsf{s}(\mathbb{Z}_n^2) \leq \frac{41}{10}n$. The Kemnitz Conjecture was finally established by Reiher [58].

For general d, Elsholtz [32] proved the following: *For an odd integer $n \geq 3$,*

$$\mathsf{s}(\mathbb{Z}_n^d) \geq \left(\frac{9}{8}\right)^{[\frac{d}{3}]}(n-1)2^d + 1.$$

This was later improved by Edel, Elsholtz, Geroldinger, Silke Kubertin, and Laurence Rackham [31].

The following very important result of Alon and Dubiner [17] says that the growth of $\mathsf{s}(\mathbb{Z}_n^d)$ is linear in n; when d is fixed and n is increasing:

Theorem 1.3. *There is an absolute constant $c > 0$ such that for all n we have,*

$$\mathsf{s}(\mathbb{Z}_n^d) \leq (cd\log_2 d)^d n.$$

Alon and Dubiner [17] conjectured that there is an absolute constant $c > 0$ such that

$$\mathsf{s}(\mathbb{Z}_n^d) \leq c^d n, \text{ for all } n \text{ and } d.$$

For other important contributions and further details related to this section, one may look into the survey article [25] of Caro, the survey article [37] of Gao and Geroldinger and the book [38] of Geroldinger and Halter-Koch.

2. Weighted Generalizations I

In this section, we discuss the following conjecture of Caro [25] and some related results.

Conjecture 2.1. (Caro) Let n and $k \geq 2$ be positive integers. Let $w_1, w_2, \ldots, w_k$ be a sequence of integers such that $\sum_1^k w_i = 0 \pmod{n}$. Then, given a sequence $x_1, x_2, \ldots, x_{n+k-1}$ of integers, $\sum_{i=1}^k w_i x_{\sigma(i)} = 0$, for some permutation σ of $[n+k-1]$.

The above conjecture is about a weighted generalization of the EGZ theorem; taking $k = n$ and $w_i = 1$ for all i, one sees that the conjecture implies the EGZ theorem.

The following result of Hamidoune [47] confirmed the above conjecture of Caro in a special case.

Theorem 2.2. *Let G be an abelian group of order n and k a positive integer. Let $(w_1, w_2, ..., w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $A : (x_1, x_2, ..., x_{k+n-1})$ of elements of G, if x_1 is the most repeated element in the sequence, we have*

$$\sum_1^k w_i x_{\sigma(i)} = \left(\sum_1^k w_i \right) x_1,$$

for some permutation σ of $[k+n-1]$.

The following result proved in [6] in the spirit of a result of Bollobás and Leader [18], implies the above theorem. A method of Yu [64], was employed in [6].

Theorem 2.3. (Adhikari, Chintamani, Moriya, Paul) *Let G be an abelian group of order n and k a positive integer. Let $(w_1, w_2, \ldots, w_k)$ be a sequence of integers where each w_i is co-prime to n. Then, given a sequence $A : (x_1, x_2, \ldots, x_{k+r})$ of elements of G, where $1 \leq r \leq n-1$, if 0 is the most repeated element in the sequence, and $\sum_{i=1}^k w_i x_{\sigma(i)} \neq 0$, for all permutations*

σ of $[k+r]$, we have

$$\left|\left\{\sum_{i=1}^{k} w_i x_{\sigma(i)} : \sigma \text{ is a permutation of } [k+r]\right\}\right| \geq r+1.$$

For further information regarding these results, we refer to a paper of Grynkiewicz [42], where, among other things, the above mentioned conjecture of Caro has been established in full generality.

3. Weighted Generalizations II

In this section, we take up a weighted version of the zero-sum constants $D(G)$ and $E(G)$ which were initiated in the papers [5], [14], [61], [4].

For a finite abelian group G of exponent m, and any non-empty subset A of $\{1, \cdots, m-1\}$, the Davenport constant of G with weight A, denoted by $D_A(G)$, is defined to be the least natural number k such that for any sequence $(x_1, \cdots, x_k)$ with $x_i \in G$, there exists a non-empty subsequence $(x_{j_1}, \cdots, x_{j_l})$ and $a_1, \cdots a_l \in A$ such that $\sum_{i=1}^{l} a_i x_{j_i} = 0$.

Similarly, one defines the constant $E_A(G)$ requiring an A-weighted zero-sum subsequence of length $|G|$.

The case $A = \{1\}$ corresponds to $D(G)$ and $E(G)$.

We write $E_A(n)$ and $D_A(n)$ respectively for $E_A(\mathbb{Z}_n)$ and $D_A(\mathbb{Z}_n)$.

We now see some results related to the weight sets considered in some early papers in this area.

- If $A = \{1, -1\}$, one has $D_A(n) \leq [\log_2 n] + 1$ by the pigeonhole principle; and considering the sequence $(1, 2, \ldots, 2^r)$, where r is defined by $2^{r+1} \leq n < 2^{r+2}$, it follows that $D_A(n) \geq [\log_2 n] + 1$. Thus $D_A(n) = [\log_2 n] + 1$.

  It was shown by Adhikari, Chen, Friedlander, Konyagin and Pappalardi [5] that
  $$E_A(n) = n + [\log_2 n].$$
- For the case $A = \{1, 2, \ldots, n-1\}$, one observes [5] that $E_A(n) = n+1$ and $D_A(n) = 2$.
- For $A = \{a \in \{1, 2, \ldots, n-1\} | (a, n) = 1\}$, it was conjectured in [5] that
  $$E_A(n) = n + \Omega(n),$$

where $\Omega(n)$ denotes the number of prime factors of n, multiplicity included.

It was first proved by Luca [50] and later independently by Griffiths [42] by a different method.

If $n = p_1 \cdots p_s$ where p_i's are primes, the sequence $(1, p_1, p_1p_2, \ldots, p_1p_2 \cdots p_{s-1})$ gives the lower bound $D_A(n) \geq 1 + \Omega(n)$. Now, $n + \Omega(n) = E_A(n) \geq D_A(n) + n - 1$ and hence $D_A(n) = 1 + \Omega(n)$.

- Considering the group $G = \mathbb{Z}_p$, for a prime p, taking $A = \{1, 2, \ldots, r\}$, where r is an integer such that $1 < r < p$, Adhikari and Rath [14] showed that

$$D_A(p) = \left\lceil \frac{p}{r} \right\rceil, \ E_A(p) = p - 1 + D_A(p).$$

- If $G = \mathbb{Z}_p$, where p is a prime, taking A to be the set of quadratic residues, following an idea of Alon, using the Chevalley-Warning theorem, it was shown by Adhikari and Rath [14] that

$$D_A(p) = 3, \ E_A(p) = p + 2.$$

Here is a sketch of the proof.

Given any sequence $(s_1, \cdots, s_{p+2})$ of elements of $\mathbb{Z}/p\mathbb{Z}$, considering the system of equations

$$\sum_{i=1}^{p+2} s_i x_i^2 = 0, \ \sum_{i=1}^{p+2} x_i^{p-1} = 0,$$

one obtains

$$E_A(p) \leq p + 2.$$

Then considering a sequence $v_1, -v_2$, where v_1 is a quadratic residue and v_2 a quadratic non-residue $\pmod p$, one has

$$3 \leq D_A(p) \leq E_A(p) - p + 1 \leq 3.$$

The above observations led ([14], [4], [61], [40]) to the following conjecture which is a weighted generalization of Theorem 1.2 of Gao.

$$E_A(G) = D_A(G) + n - 1. \tag{3.1}$$

For $G = \mathbb{Z}/p\mathbb{Z}$, p prime, it was observed by Adhikari and Rath [14] by using combinatorial methods and results from additive combinatorics and later a proof was obtained by Adhikari, Gun and Rath [9] by using the permanent lemma. The result also follows from the following more general result of Adhikari and Chen [4].

Let G be a finite abelian group of order n, and $A = \{a_1, a_2, \cdots, a_r\}$ be a finite subset of $\mathbb{Z}$ with $|A| = r \geq 2$ and $\gcd(a_2 - a_1, a_3 - a_1, \cdots, a_r - a_1, n) = 1$. Then we have $E_A(G) = D_A(G) + n - 1$.

The conjecture was established for cyclic groups by Yuan and Zeng [65] and for general finite abelian groups, it was established by Grynkiewicz, Marchan and Ordaz [45] (one may also see [43]). An important ingredient in both these proofs is a result of Devos, Goddyn and Mohar [30].

We would like to remark that most often, $D_A(G)$ is easier to be computed and from the relation (3.1) one can obtain the value of the corresponding $E_A(G)$. For example, when $A \subseteq (\mathbb{Z}/p\mathbb{Z})^*$ such that $|A| \geq \frac{p+2}{3}$, and there exists α in $(\mathbb{Z}/p\mathbb{Z})^*$ such that $\frac{a}{b} \neq \alpha$ for any a, b in A, then Adhikari, Balasubramanian, Pappalardi and Rath [3] showed that $D_A(p) = 3$, and hence one obtains that $E_A(p) = p + 2$. Many people have worked (see [7], [27], [41], [62], [63], [44], [10], for instance) in the problem of finding exact values and good bounds for these constants. We do not go into them here.

4. Weighted Generalizations III

We now take up the weighted version of $s(G)$ and Harborth's constant.

For a nonempty subset A of $[1, \exp(G) - 1] = \{1, \ldots, \exp(G) - 1\}$, we define $s_A(G)$ to be the least integer k such that any sequence S with length $|S| \geq k$ of elements in G has an A-weighted zero-sum subsequence of length $\exp(G)$.

Study of the weighted version of Harborth's constant $\mathsf{s}_A(C_n^d)$ was initiated in a paper of Adhikari, Balasubramanian, Pappalardi and Rath [3].

We state some preliminary results for $\mathsf{s}_A(\mathbb{Z}_n^d)$:

- If the weight set A is closed under multiplication, then recovering sequences from quotients becomes possible and similar to Harborth's, one obtains the following upper bound :

$$\begin{aligned} \mathsf{s}_A(\mathbb{Z}_{mn}^d) \quad &\leq \quad \min(\mathsf{s}_A(\mathbb{Z}_n^d) + n(\mathsf{s}_A(Z_m^d) - 1), \\ &\qquad \mathsf{s}_A(\mathbb{Z}_m^d) + m(\mathsf{s}_A(\mathbb{Z}_n^d) - 1)). \end{aligned}$$

- For the weight set $A = \{1, -1\}$, Adhikari, Balasubramanian, Pappalardi and Rath [3] observed the following.

For an odd integer n, we have

$$\mathsf{s}_{\{1,-1\}}(\mathbb{Z}_n^2) = 2n - 1.$$

We give some more results on $s_A(G)$ and $\mathsf{s}_A(\mathbb{Z}_n^d)$:

- Adhikari, Ambily and Sury [2] observed the following.

  Let p be a prime and $A \subset [1, p-1]$. If $\{a \pmod p : a \in A\}$ is a subgroup of the multiplicative group $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$, then

  $$\mathsf{s}_A(\mathbb{Z}_p^r) \leq \frac{r(p-1)}{|A|} + p, \text{ for } 1 \leq r < \frac{p|A|}{p-1}.$$

  In particular, $\mathsf{s}_A(\mathbb{Z}_p^{|A|}) \leq 2p - 1$.
- The following is an essentially sharp upper bound by Adhikari, Grynkiewicz and Sun [8], where one does not have the restriction that the weight set A is a group.

  Let p be a prime and let G be an abelian p-group with $|G| > 1$ and exponent p^{k_r}. Let $\emptyset \neq A \subset [1, p^{k_r} - 1] \setminus p\mathbb{Z}$ and suppose that any two distinct elements of A are incongruent modulo p.

  Then, for each $k \in \mathbb{Z}^+$, any sequence of elements of G of length at least $p^k - 1 + \lceil (d(G))/|A| \rceil$ contains a nonempty A-weighted zero-sum subsequence whose length is divisible by p^k.

  Thus, if $|A|$ $(\exp(G) - 1) \geq D(G) - 1$ (which happens if $|A|$ is at least r, the rank of G), then we have

  $$\mathsf{s}_A(G) \leq \exp(G) - 1 + \left\lceil \frac{D(G)}{|A|} \right\rceil.$$
- The following result of Adhikari, Grynkiewicz and Sun [8] gives the asymptotic behavior of $\mathsf{s}_{\{\pm 1\}}(G)$, for a finite abelian group G of even exponent.

  For finite abelian groups of even exponent and fixed rank,

  $$\mathsf{s}_{\{\pm 1\}}(G) = \exp(G) + \log_2 |G| + O(\log_2 \log_2 |G|),$$

  as $\exp(G) \to \infty$.
- By suitable modifications of the polynomial method of Rónyai, some results corresponding to odd exponents and rank 3 were obtained for some weight sets in some papers of Adhikari, Mazumdar, Roy, Sarkar and Hegde (see [12], [13], [15], [11]) .

5. Final Remarks

Like their classical counterparts, the study of the weighted generalizations involves combinatorial methods, applications of results in set additions and polynomial methods; and, apart from the fact that they arise rather naturally, one has already seen some applications of these.

Applications of weighted zero-sum constants have been made in Algebraic Number Theory [46] and interactions with coding theory have been observed in a paper of Marchan, Ordaz, Santos, and Schmid [51].

For instance, Halter-Koch has shown [46] that the plus-minus weighted Davenport constant is related to questions on the norms of principal ideals in quadratic number fields. More precisely, he has proved the following.

Let K be a quadratic algebraic number field, $\mathcal{O}_K$ its ring of integers and $\mathcal{C}_K$ its ideal class group. Then $D_{\{\pm 1\}}(\mathcal{C}_K)$ is the smallest positive integer l with the following property:

If $q_1, q_2, \dots q_l$ are pairwise co-prime positive integers such that their product $q = q_1 \cdots q_l$ is the norm of an ideal of $\mathcal{O}_K$, then some divisor $d > 1$ of q is the norm of a principal ideal of $\mathcal{O}_K$.

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A WALK THROUGH IRREDUCIBLE POLYNOMIALS*

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Abstract. Criteria for irreducibility of polynomials have a long history. In 1797, Gauss proved that the only irreducible polynomials (in one variable) with complex coefficients are linear polynomials. However, for polynomials with rational coefficients, the Eisenstien Irreducibility Criterion, proved in 1850, implies that for each positive integer n, there are infinitely many irreducible polynomials of degree n. In this paper, we shall discuss some generalizations, discovered in recent years using the theory of valuations, of the classical irreducibility criteria of Eisenstein, Schönemann and Dumas.

1. Eisenstein Irreducibility Criterion

The most famous irreducibility criterion for polynomials with coefficients in the ring $\mathbb{Z}$ of integers is the one proved by Eisenstein [3] in 1850. It states as follows.

Theorem 1.1. (Eisenstein Irreducibility Criterion) *Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with coefficients in the ring $\mathbb{Z}$ of integers. Suppose that there exists a prime number p such that a_n is not divisible by p, a_i is divisible by p for $0 \leq i \leq n-1$, and a_0 is not divisible by p^2. Then $f(x)$ is irreducible over the field $\mathbb{Q}$ of rational numbers.*

Proof. Assume to the contrary that $f(x)$ is reducible over $\mathbb{Q}$ and hence can be written as a product of two non-constant polynomials with coefficients in $\mathbb{Z}$ by Gauss Lemma[1]. Write

$$f(x) = g(x)h(x) = (b_0 + b_1x + \cdots + b_rx^r)(c_0 + c_1x + \cdots + c_sx^s).$$

We have $a_i = b_ic_0 + b_{i-1}c_1 + \cdots + b_0c_i$. Since $p|a_0$ and $p^2 \nmid a_0$ and $a_0 = b_0c_0$, we deduce $p|b_0$ or $p|c_0$ but not both. Suppose that $p|b_0$ and $p \nmid c_0$. Since

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[1]Gauss Lemma states that product of two primitive polynomials in $\mathbb{Z}[x]$ is primitive.

$p \nmid a_n$, it follows that p does not divide b_i for some i. Choose least i such that p does not divide b_i. Since $i \leq r < n$, by hypothesis p divides $a_i = b_i c_0 + b_{i-1} c_1 + \cdots + b_0 c_i$; as p divides $b_0, b_1, \ldots, b_{i-1}$, we conclude that p divides $b_i c_0$. This is a contradiction because p does not divide b_i, c_0. Hence $f(x)$ is irreducible over $\mathbb{Q}$. $\square$

Definition 1.2. A polynomial which satisfies the three conditions mentioned in Theorem 1.1 is called an *Eisenstein polynomial* with respect to the prime p.

Example 1.3. The polynomial $x^7 + 4x + 2$ is irreducible over $\mathbb{Q}$.

Example 1.4. For a prime p, consider the p^{th} cyclotomic polynomial

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = \frac{x^p - 1}{x - 1}.$$

The polynomial

$$\begin{aligned} \Phi_p(x+1) &= \frac{(x+1)^p - 1}{x} \\ &= x^{p-1} + \binom{p}{1} x^{p-2} + \cdots + \binom{p}{p-2} x + \binom{p}{p-1} \end{aligned}$$

is irreducible over $\mathbb{Q}$ by Eisenstein Irreducibility Criterion. So is $\Phi_p(x)$.

2. Schönemann Irreducibility Criterion and its Generalization

In 1846, Schönemann [9] proved the following result.

Theorem 2.1. (Schönemann Irreducibility Criterion) *Let $\phi(x)$ belonging to $\mathbb{Z}[x]$ be a monic polynomial which is irreducible modulo a given prime p. Let $f(x)$ belonging to $\mathbb{Z}[x]$ be of the form $f(x) = \phi(x)^s + pM(x)$ where $M[x] \in \mathbb{Z}[x]$ has degree less than $s \deg \phi(x)$. If $\phi(x)$ is coprime to $M(x)$ modulo p, then $f(x)$ is irreducible in $\mathbb{Q}[x]$.*

Proof. For $g(x) \in \mathbb{Z}[x]$, let $\bar{g}(x)$ denote the polynomial obtained by replacing each coefficient of $g(x)$ modulo p. Suppose to the contrary $f(x)$ is reducible over $\mathbb{Q}$. Then by Gauss Lemma, $f(x) = g(x)h(x)$ where $g(x), h(x)$ are monic polynomials belonging to $\mathbb{Z}[x]$ of positive degree. Consequently $\bar{f}(x) = \bar{g}(x)\bar{h}(x)$. In view of the hypothesis, $\bar{f}(x) = \bar{\phi}(x)^s$. Since $\bar{\phi}(x)$ is irreducible over $\mathbb{Z}/p\mathbb{Z}$, it follows that $\bar{g}(x) = \bar{\phi}(x)^d$, $\bar{h}(x) = \bar{\phi}(x)^e$ for some positive integers d, e. Therefore we can write

$$g(x) = \phi(x)^d + pg_1(x), \quad h(x) = \phi(x)^e + ph_1(x)$$

where $g_1(x), h_1(x)$ belonging to $\mathbb{Z}[x]$ have degree less than $d \deg \phi(x), e \deg \phi(x)$ respectively. On multiplying, we see that

$$f(x) = g(x)h(x) = \phi(x)^s + p[\phi(x)^d h_1(x) + \phi(x)^e g_1(x) + pg_1(x)h_1(x)].$$

By hypothesis, $f(x) = \phi(x)^s + pM(x)$. The above equation shows that $\bar{M}(x) = \bar{\phi}(x)^d \bar{h}_1(x) + \bar{\phi}(x)^e \bar{g}_1(x)$ is divisible by $\bar{\phi}(x)$. This contradicts the hypothesis that $\bar{\phi}(x)$ does not divide $\bar{M}(x)$ and hence the theorem is proved. $\square$

Eisenstein's Criterion is easily seen to be a particular case of Schönemann Criterion by setting $\phi(x) = x$. We now restate the latter criterion using $\phi(x)$-expansion defined below.

Definition 2.2. If $\phi(x)$ is a fixed monic polynomial with coefficients from an integral domain R, then each $f(x) \in R[x]$ can be uniquely written as $\sum\limits_i A_i(x)\phi(x)^i$ with $\deg A_i(x) < \deg \phi(x)$ $\forall i$; this expansion is referred to as the $\phi(x)$-*expansion* of $f(x)$.

Theorem 2.3 (**Restatement of Schönemann Irreducibility Criterion**)**.** *Let $\phi(x)$ belonging to $\mathbb{Z}[x]$ be a monic polynomial which is irreducible modulo a given prime p. Let $f(x)$ belonging to $\mathbb{Z}[x]$ be a monic polynomial having $\phi(x)$-expansion $\sum\limits_{i=0}^{s} A_i(x)\phi(x)^i$ with $A_s(x) = 1$. If p divides the content of $A_i(x)$ for $0 \leq i < s$ and p^2 does not divide the content of $A_0(x)$, then $f(x)$ is irreducible over $\mathbb{Q}$.*

Example 2.4. The polynomial $(x^2+1)^3 + 3(b_1x+b_2)(x^2+1)^2 + 3(c_1x+c_2)(x^2+1) + 3(d_1x+d_2)$ is irreducible over $\mathbb{Q}$ for all integers $b_1, b_2, c_1, c_2, d_1, d_2$ with 3 not dividing at least one of d_1, d_2 in view of Schönemann Irreducibility Criterion.

Recall that the content of a polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ with coefficients in $\mathbb{Z}$ is defined to be the gcd of $a_0, a_1, \ldots, a_n$; it will be denoted be $c(f)$. In 2017, A. Jakhar and N. Sangwan [5] extended Theorem 2.3 in the following form.

Theorem 2.5. *Let p be a prime number and $\phi(x) \in \mathbb{Z}[x]$ be a monic polynomial which is irreducible modulo p. Let $f(x)$ belonging to $\mathbb{Z}[x]$ be a polynomial having $\phi(x)$-expansion $A_s(x)\phi(x)^s + \cdots + A_1(x)\phi(x) + A_0(x)$, where $A_i(x) \in \mathbb{Z}[x]$, $p \nmid c(A_s(x))$, $p|c(A_i(x))$ for $0 \leq i < s$. Assume that p^2 does not divide $c(A_i(x))$ for at least one i, $0 \leq i < s$, and let $k < s$ be the smallest non-negative integer such that $p^2 \nmid c(A_k(x))$. If $f(x) = g(x)h(x)$ with $g(x), h(x) \in \mathbb{Z}[x]$, then*

$$\min\{\deg(g(x)), \deg(h(x))\} \leq \deg(A_s(x)) + k \deg(\phi(x)).$$

Proof. For a polynomial $t(x) \in \mathbb{Z}[x]$, let $\bar{t}(x)$ denote the polynomial obtained by reducing its coefficients modulo p. Note that $\bar{f}(x) = \bar{g}(x)\bar{h}(x) = \overline{A}_s(x)(\overline{\phi}(x))^s$, and

$$\deg(g(x)) + \deg(h(x)) = \deg(A_s(x)) + s(\deg(\phi(x))). \tag{2.1}$$

Let $\sum_i B_i(x)(\phi(x))^i, \sum_j C_j(x)(\phi(x))^j$ be the $\phi(x)$-expansions of polynomials $g(x)$, $h(x)$ respectively. Let $d, e \geq 0$ be the smallest indices in the $\phi(x)$-expansions of $g(x)$, $h(x)$ respectively such that $p \nmid c(B_d(x)), p \nmid c(C_e(x))$ and $d', e' \geq 0$ the largest indices in the $\phi(x)$-expansions of $g(x), h(x)$ respectively such that $p \nmid c(B_{d'}(x)), p \nmid c(C_{e'}(x))$. Therefore,

$$g(x) = B_{d'}(x)(\phi(x))^{d'} + \cdots + B_d(x)(\phi(x))^d + pG(x)$$

for some $G(x) \in \mathbb{Z}[x]$.

Claim is that $d' = d$. If $d' > d$ then $\bar{g}(x)$ would be divisible by a polynomial of degree greater than or equal to $(d' - d)\deg(\phi(x))$ which is coprime to $\overline{\phi}(x)$; this is impossible because $\bar{g}(x)\bar{h}(x) = \overline{A}_s(x)(\overline{\phi}(x))^s$ and $\deg(A_s(x)) < \deg(\phi(x))$, hence $d' = d$. Similarly $e' = e$. So we can write

$$g(x) = B_d(x)(\phi(x))^d + pG(x), \tag{2.2}$$

$$h(x) = C_e(x)(\phi(x))^e + pH(x) \tag{2.3}$$

for some $H(x) \in \mathbb{Z}[x]$. Since $\bar{f}(x) = \bar{g}(x)\bar{h}(x) = \overline{A}_s(x)(\overline{\phi}(x))^s$, it follows from the above two equations that $s = d + e$. Using (2.1), (2.3) and the fact that $s = d + e$, we see that

$$\deg(g(x)) \leq \deg(A_s(x)) + d(\deg(\phi(x))). \tag{2.4}$$

Arguing similarly, it can be seen that

$$\deg(h(x)) \leq \deg(A_s(x)) + e(\deg(\phi(x))). \tag{2.5}$$

Multiplying (2.2) and (2.3), we have

$$f(x) = B_d(x)C_e(x)\phi(x)^{d+e} + p[H(x)B_d(x)\phi(x)^d + G(x)C_e(x)\phi(x)^e] + p^2G(x)H(x).$$

Keeping in mind $s = d + e$ and definition of k, it follows from the last equation that

$$k(\deg(\phi(x)) \geq \min\{d(\deg(\phi(x))), e(\deg(\phi(x)))\}. \tag{2.6}$$

Adding $\deg(A_s(x))$ to both sides of the above equation and using (2.4), (2.5), we see that

$$\deg(A_s(x)) + k\deg(\phi(x)) \geq \min\{\deg(g(x)), \deg(h(x))\}.$$

□

Remark 2.6. In the above theorem, if $k = 0$ and $\deg(A_s(x)) = 0$, then this yields the classical Schönemann Irreducibility Criterion. Also, if $k = 0$, $\deg(A_s(x)) = 1$ or $k = 1$, $\deg(A_s(x)) = 0$, $\deg(\phi(x)) = 1$, then either $f(x)$ has a linear factor over $\mathbb{Q}$ or $f(x)$ is irreducible over $\mathbb{Q}$.

Example 2.7. Let $f(x) = x(x^2 + 3)^3 + 5(x^2 + 3)^2 + 5(x^2 + 3) + 5$. Then taking $\phi(x) = x^2 + 3$ and applying Theorem 2.5 with $p = 5$, we see that either $f(x)$ has a linear factor over $\mathbb{Q}$ or it is irreducible over $\mathbb{Q}$. It can be easily checked using rational root theorem that $f(x)$ does not have a linear factor over $\mathbb{Q}$. Therefore $f(x)$ is irreducible over $\mathbb{Q}$

The following theorem which extends Eisenstein Irreducibility Criterion was first proved by Weintraub [10] in 2013. It is an immediate consequence of Theorem 2.5 on taking $\phi(x) = x$.

Theorem 2.8. *Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_0$ belonging to $\mathbb{Z}[x]$ be a polynomial. Suppose there is a prime p such that p does not divide a_n, p divides a_i for $i = 0, \cdots, n-1$ and for some k with $0 \leq k \leq n-1, p^2$ does not divide a_k. Let k_0 be the smallest such value of k. If $f(x) = g(x)h(x)$ is a factorization in $\mathbb{Z}[x]$, then* $\min(\deg g(x), \deg h(x)) \leq k_0$.

Notation. Let p be a prime number. In what follows for any non-zero integer c, $v_p(c)$ will denote the highest power of p dividing c. Set $v_p(0) = \infty$. The map v_p satisfies the following properties for all $a, b \in \mathbb{Z}$:

(i) $v_p(ab) = v_p(a) + v_p(b)$.

(ii) $v_p(a+b) \geq \min\{v_p(a), v_p(b)\}$.

The map v_p is called the p-adic valuation[2].

Observation. Theorem 2.8 is significant only when $k_0 < \dfrac{n}{2}$. Its hypothesis implies that the integer $k_0 < \dfrac{n}{2}$ is characterized by the property that it is the smallest index for which

$$\min_{0\leq i\leq n-1}\left\{\frac{v_p(a_i)}{n-i}\right\} = \frac{1}{n-k_0},$$

because if $0 \leq i < k_0$, then $\dfrac{v_p(a_i)}{n-i} \geq \dfrac{2}{n-i} > \dfrac{1}{n-k_0}$.

In view of the above observation, the next theorem which was proved by B. Jhorar and S.K. Khanduja in [6] extends Theorem 2.8; indeed they have proved the above theorem in more general setup replacing v_p by an arbitrary valuation of a field.

Theorem 2.9. *Let p be a prime number. Let $f(x) = \sum\limits_{i=0}^{n} a_i x^i$ be a polynomial having coefficients in $\mathbb{Z}$ with p not dividing a_n. Let k be the smallest index such that* $\min\left\{\dfrac{v_p(a_i)}{n-i} | 0 \leq i \leq n-1\right\} = \dfrac{v_p(a_k)}{n-k}$. *If $v_p(a_k)$ is co-prime with $n-k$, then for any factorization of $f(x)$ as $g(x)h(x)$ in $\mathbb{Z}[x]$, we have* $\min\{\deg g(x), \deg h(x)\} \leq k$.

Example 2.10. Let $f(x) = x^4 + pb_3x^3 + p^2b_2x^2 + p^2b_1x + p^3b_0$ with $p \nmid b_1$ be a polynomial with integer coefficients. Then by Theorem 2.9, either $f(x)$ has a linear factor over $\mathbb{Q}$ or $f(x)$ is irreducible over $\mathbb{Q}$. In particular, $x^4+4x+8m$ is irreducible over $\mathbb{Q}$ for all $m \in \mathbb{N}$. Note that the irreducibility of polynomial given in this example can not be proved using Theorem 2.8. So Theorem 2.9 is stronger that Theorem 2.8.

3. An Extension of Dumas Irreduciblity Criterion

In 1906, Dumas [2] generalised Eisenstein Irreducibility Criterion and proved the following result.

Theorem 3.1 (**Classical Dumas Irreducibility Criterion**). *Let* $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ *be a polynomial with coefficients from $\mathbb{Z}$. Suppose*

[2]p-adic valuation was first defined by Kurt Hensel.

that there exists a prime p such that $v_p(a_n) = 0$, $nv_p(a_i) \geq (n-i)v_p(a_0)$ for $1 \leq i \leq n-1$ and $v_p(a_0)$ is coprime to n, then $f(x)$ is irreducible over $\mathbb{Q}$.

Note that Eisenstein Irreducibility Criterion is a special case of the above criterion with $v_p(a_0) = 1$. It may be pointed out that Dumas Irreducibility Criterion is stronger than Eisenstein Irreducibility Criterion because the polynomial $x^3 + 3x^2 + 9x + 9$ is irreducible over $\mathbb{Q}$ in view of Dumas Irreducibility Criterion whereas the latter one is not applicable.

We shall prove Dumas criterion in a more general setup for polynomials with coefficients in arbitrary valued fields as defined below.

Definition 3.2. By a Krull valuation v of a field K we mean a mapping v from K onto $G \cup \{\infty\}$, where G is a totally ordered (additively written) abelian group, such that for all a, b in K, the following hold:

(i) $v(a) = \infty$ if and only if $a = 0$;

(ii) $v(ab) = v(a) + v(b)$;

(iii) $v(a + b) \geq \min\{v(a), v(b)\}$.

The set $R_v = \{a \in K \mid v(a) \geq 0\}$ is a subring of K, called the valuation ring of v, with unique maximal ideal m_v given by $m_v = \{a \in K \mid v(a) > 0\}$. R_v/m_v is called the residue field of v and G the value group of v.

A real valuation is said to be *discrete* if the additive group $v(K^\times)$ is infinite cyclic group, i.e., isomorphic to $(\mathbb{Z}, +)$. It can be easily seen that the valuation ring of a discrete valuation ring is a Principal Ideal Domain. In this case, a generator α of the ideal M_v is a prime element of R_v with $v(\alpha)$ being the smallest positive element in $v(K^\times)$.

Note that if the field K is the quotient field of an integral domain R and v is a function on R satisfying properties (i), (ii) and (iii) mentioned in the above definition, then v can be extended uniquely to a valuation of v in a natural way. The unique extension of v_p to $\mathbb{Q}$ is called the p-adic valuation of $\mathbb{Q}$.

Example 3.3. Let R be a Unique Factorisation Domain with quotient field K and π be a prime element of R. We denote the π-adic valuation of K defined for any non-zero $\alpha \in R$ by $v_\pi(\alpha) = m$, if $\alpha = \pi^m\beta$ for some $\beta \in R$, with π not dividing β. It can be extended to K in a canonical manner.

Example 3.4 (**Example of Krull valuation**). The ring $\mathbb{Q}[x]$ of polynomials in an indeterminate x is a Unique Factorisation Domain. Let v_x denote the x-adic valuation of its quotient field $\mathbb{Q}(x)$ corresponding to the irreducible element x of $\mathbb{Q}[x]$. For any non-zero polynomial $g(x)$ belonging to $\mathbb{Q}[x]$, we shall denote by g^* the constant term of the polynomial $g(x)/x^{v_x(g(x))}$. Let p be any rational prime. Let v be the mapping from non-zero elements of $\mathbb{Q}(x)$ to $\mathbb{Z} \times \mathbb{Z}$ (lexicographically ordered) defined on $\mathbb{Q}[x]$ by

$$v(g(x)) = (v_x(g(x)), v_p(g^*)).$$

Then v gives a valuation on $\mathbb{Q}(x)$.

The result proved below will be used in the sequel.

Theorem 3.5 (**Strong Triangle Law**). *Let v be a valuation of a field K. If $\alpha, \beta \in K$ are such that $v(\alpha) \neq v(\beta)$, then $v(\alpha + \beta) = \min\{v(\alpha), v(\beta)\}$.*

Proof. Assume without loss of generality that $v(\alpha) < v(\beta)$. By the definition of valuation

$$v(\alpha + \beta) \geq \min\{v(\alpha), v(\beta)\} = v(\alpha). \tag{3.1}$$

Again by a defining property of valuation,

$$v(\alpha) = v(\alpha + \beta - \beta) \geq \min\{v(\alpha + \beta), v(-\beta)\}.$$

Since $2v(-1) = v(1) = 0$, we have $v(-\beta) = v(\beta)$. So the above minimum has to be $v(\alpha + \beta)$ in view of the assumption $v(\alpha) < v(\beta)$. It now follows from (3.1) that $v(\alpha + \beta) = v(\alpha)$ as desired. □

It may be pointed out that the proofs of Eisenstein and Schönemann Irreducibility Criteria can be carried over to polynomials with coefficients in the valuation ring of a discrete valuation v on replacing p by a prime element of R_v. Some generalisations of these criteria for polynomials over valued fields are also known (cf. [1], [6], [7], [8]).

The following proposition will be used in the proof of Dumas Irreducibility Criterion.

Proposition 3.6. *Let v be a valuation of a field K with value group G, μ be an element of a totally ordered abelian group containing G as an ordered subgroup and let $w : K[x] \longrightarrow G \cup \{\infty\}$ be the mapping defined by*

$$w\Big(\sum_i c_i x^i\Big) = \min_i\{v(c_i) + i\mu\},\ c_i \in K.$$

Then w gives rise to a valuation on $K(x)$ whose restriction to K is v and whose value group is the subgroup of $\mathbb{R}$ generated by the value group of v and μ.

Proof. Observe that $w(f(x)) = \infty$ if and only if $f(x) = 0$. It will now be shown that if $f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{m} b_j x^j$ are polynomials in $K[x]$, then

$$w(fg) = w(f) + w(g),\ w(f+g) \geq \min\{w(f), w(g)\}.$$

Write $fg = \sum_{k=0}^{m+n} c_k x^k$ where $c_k = \sum_{i+j=k} a_i b_j$. Let i_o, j_o be chosen so that

$$i_o = \min\{i \mid v(a_i) + i\mu = w(f)\},\ j_o = \min\{j \mid v(b_j) + j\mu = w(g)\}.$$

Then

$$c_{i_o+j_o} = a_{i_o} b_{j_o} + \sum_{i+j=i_o+j_o, i\neq i_o} a_i b_j. \tag{3.2}$$

We show that

$$v(c_{i_o+j_o}) = v(a_{i_o} b_{j_o}).$$

Since $i \neq i_o, i+j = i_o + j_o$ imply that either $i < i_o$ or $j < j_o$, so either $v(a_{i_o}) + i_o\mu < v(a_i) + i\mu$ or $v(b_{j_o}) + j_o\mu < v(b_j) + j\mu$. Thus $v(a_{i_o}) + i_o\mu + v(b_{j_o}) + j_o\mu < v(a_i) + i\mu + v(b_j) + j\mu$ *when* $i + j = i_o + j_o, i \neq i_o$. Consequently with $i+j = i_o+j_o, i \neq i_o$, we have $v(a_{i_o} b_{j_o}) < v(a_i b_j)$. Hence by (3.2) and strong triangle law, we have

$$v(c_{i_o+j_o}) = v(a_{i_o} b_{j_o}).$$

Therefore

$$v(c_{i_o+j_o}) + (i_o + j_o)\mu = v(a_{i_o} b_{j_o}) + (i_o + j_o)\mu = w(f) + w(g).$$

Thus we have shown that

$$w(fg) \leq v(c_{i_o+j_o}) + (i_o + j_o)\mu = w(f) + w(g). \tag{3.3}$$

On the other hand, for any $k, 0 \leq k \leq m+n$,

$$\begin{aligned} v(c_k) + k\mu = v(\sum_{i+j=k} a_i b_j) + k\mu &\geq \min_{i,j}\{v(a_i) + v(b_j) \mid i+j=k\} + k\mu \\ &= \min_{i,j}\{(v(a_i) + i\mu) + (v(b_j) + j\mu) \mid i+j=k\} \\ &\geq w(f) + w(g). \end{aligned}$$

So

$$w(fg) \geq w(f) + w(g). \tag{3.4}$$

By (3.3) and (3.4), we have $w(fg) = w(f) + w(g)$.

It remains to verify that $w(f+g) \geq \min\{w(f), w(g)\}$. Assume without loss of generality that $n = \max\{\deg f, \deg g\}$. Set $b_i = 0$ if $m+1 \leq i \leq n$. Then for any i, $0 \leq i \leq n$, we have

$$\begin{aligned} v(a_i + b_i) + i\mu &\geq \min\{v(a_i), v(b_i)\} + i\mu \\ &= \min\{(v(a_i) + i\mu, v(b_i) + i\mu)\} \\ &\geq \min\{w(f), w(g)\}. \end{aligned}$$

Therefore

$$w(f+g) \geq \min\{w(f), w(g)\}.$$

□

We now prove the following theorem which has been proved in 2020 by A .Jakhar with slightly stronger hypothesis that the valuation of each coefficient except the leading coefficient of the polynomial is positive (cf. [4]). This theorem immediately yields Dumas Irreducibility Criterion.

Theorem 3.7. *Let v be a valuation of a field K with value group G. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, a_0 \neq 0$ be a polynomial having coefficients in K with $v(a_n) = 0$ and $nv(a_i) \geq (n-i)v(a_0)$ for $0 \leq i \leq n-1$. Let d be the smallest positive integer such that $d\frac{v(a_0)}{n} \in G$. Then each irreducible factor of $f(x)$ over K has degree at least d.*

Proof. Set $\mu = v(a_0)/n$. Let w denote the mapping on $K[x]$ defined by

$$w(\sum_i c_i x^i) = \min_i\{v(c_i) + i\mu\},\ c_i \in K.$$

By Proposition 3.6, w gives a valuation on $K[x]$. In view of the hypothesis, we see that $\mu = \frac{v(a_0)}{n} = \min\limits_{0 \leq i \leq n-1} \left\{\frac{v(a_i)}{n-i}\right\}$. Therefore

$$w(f(x)) = \min_{0 \leq i \leq n}\{v(a_i) + i\mu\} = v(a_0) = n\mu. \tag{3.5}$$

Let $f(x) = f_1(x) f_2(x) \cdots f_t(x)$ be the factorization of $f(x)$ into irreducible factors over K where $f_i(x) = \sum\limits_{j=0}^{d_i} b_{ij} x^j$ has degree d_i and leading coefficient b_{id_i}. Since $v(a_n) = 0$, we may assume that $v(b_{id_i}) = 0$ for $1 \leq i \leq t$.

Observe that

$$v(a_0) = v(\prod_{i=1}^{t} b_{i0}) = v(b_{10}) + \cdots + v(b_{t0})$$

and $n = d_1 + \cdots + d_t$. Therefore using (3.5), we see that

$$w(f(x)) = v(a_0) = v(b_{10}) + \cdots + v(b_{t0}) = n\mu = d_1\mu + \cdots + d_t\mu. \quad (3.6)$$

Also by definition of w, we have $w(f_i(x)) = \min\limits_{0\le j\le d_i}\{v(b_{ij}) + j\mu\}$; consequently

$$w(f_i(x)) \le v(b_{i0}),\ w(f_i(x)) \le v(b_{id_i}) + d_i\mu = d_i\mu. \quad (3.7)$$

Now using (3.6), (3.7) and keeping in mind that

$$w(f(x)) = w(f_1(x)) + \cdots + w(f_t(x)),$$

it follows that

$$w(f_i(x)) = v(b_{i0}) = d_i\mu,\ 1 \le i \le t.$$

Consequently $d_i\mu \in G$ for $1 \le i \le t$. By hypothesis, d is the smallest positive element such that $d\mu \in G$, hence $d_i \ge d$ for $1 \le i \le t$. This completes the proof of the theorem. □

The next corollary which extends Dumas Irreducibility Criterion is an immediate consequence of the above theorem.

Corollary 3.8. *Let v be a valuation of a field K having value group $\mathbb{Z}$. Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0, a_0 \ne 0$ be a polynomial having coefficients in K with $v(a_n) = 0$ and $v(a_i)n \ge v(a_0)(n-i)$ for $1 \le i \le n-1$. The following hold:*

(i) *If n, $v(a_0)$ are coprime, then $f(x)$ is irreducible over K.*
(ii) *If $gcd(n, v(a_0)) = 2$, then either $f(x)$ is irreducible over K or it is a product of two irreducible polynomials of degree $\frac{n}{2}$ over K.*

The following corollary is an application of Theorem 3.7. The irreducibility of the class of polynomials occurring in this corollary cannot be established by Dumas Irreducibility Criterion.

Corollary 3.9. *Let p be a prime number and a be an integer with $v_p(a)$ positive and even. Then the polynomial $f(x) = x^6 + ax + p^2$ is irreducible over $\mathbb{Q}$.*

Proof. Suppose to the contrary that $f(x)$ is reducible over $\mathbb{Q}$. Then by assertion (ii) of Corollary 3.8, $f(x)$ factors as a product of two monic irreducible polynomials, say $g(x), g_1(x)$ of degree 3 over $\mathbb{Q}$. Since $f(x) \in \mathbb{Z}[x]$ is monic, using Gauss's lemma for primitive polynomials, it can be easily seen that $g(x), g_1(x)$ are in $\mathbb{Z}[x]$. Write $g(x) = x^3 + bx^2 + cx + d, g_1(x) = x^3 + b_1x^2 + c_1x + d_1$. On comparing coefficients in the equation $f(x) = g(x)g_1(x)$, we see that

$$dd_1 = p^2, \tag{3.8}$$

$$cd_1 + c_1d = a, \tag{3.9}$$

$$bd_1 + b_1d + cc_1 = 0, \tag{3.10}$$

$$c + c_1 + bb_1 = 0, \tag{3.11}$$

$$b + b_1 = 0. \tag{3.12}$$

Since $f(x) \equiv x^6 \pmod p$, each of b, b_1, c, c_1, d, d_1 is divisible by p. So (3.8) implies that

$$d = d_1 = \pm p. \tag{3.13}$$

Hence (3.9) shows that $c + c_1 = \pm(a/p)$. It follows from (3.10), (3.12) and (3.13) that $cc_1 = 0$, say $c_1 = 0$. Using (3.11) and (3.12), we have $\pm(a/p) = c = b^2$, which is impossible as $v_p(a/p)$ is odd. This contradiction proves that $f(x)$ is irreducible over $\mathbb{Q}$. □

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PERTURBATION OF SEMI-WEAKLY m-HYPONORMAL WEIGHTED SHIFTS

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Abstract. Semi m-hyponormality is defined and it is proved that a perturbed semi m-hyponormal weighted shift operator remains semi-weak m-hyponormality.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space and T be a bounded linear operator on H. We denote $[A, B] := AB - BA$ for the commutator of two operators A and B. An operator T is *hyponormal* if $[T^*, T] \geq 0$. An operator T is *polynomially hyponormal* if $p(T)$ is hyponormal for all (complex) polynomials p [3]. An operator T is *weakly m-hyponormal* if $p(T)$ is hyponormal for any polynomial p with degree $\leq m$ [7]. An operator T is (*strongly*) *m-hyponormal* whenever the $m \times m$ operator matrix $([T^{*j}, T^i])_{i,j=1}^m$ is positive [4]. It is well known that m-hyponormality implies weak m-hyponormality but the converse is not true in general [1, 5, 9, 10].

Curto in [5] began to study characterization of m-hyponormality within the class of weakly m-hyponormal operators to understand the gap between subnormality and hyponormality. The study of this gap has been only partially successful. McCullough and Paulsen in [11] proved that there exists a non subnormal polynomially hyponormal operator if and only if there exists a weighted shift operator with the same property. Also in [3], Curto and Putinar proved the existence of polynomially hyponormal operator which is not subnormal. This means that there must also exist a non subnormal polynomially hyponormal weighted shift operator. However, a

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concrete model of such a weighted shift operator has not yet been found. Such weighted shift operator can easily be constructed if we can answer positively the stability of polynomially hyponormal (or weakly m-hyponormal) weighted shift operators under a small perturbation of the weight sequence. The theory of perturbation of weighted shift operator was introduced by Curto in [4] and it was shown in [4, Theorem 2.3], if W_α is 2-hyponormal then W_α remains quadratically hyponormal under a small finite rank perturbation of α. However, it has not yet been known that [4, Question 5.3] if W_α is m-hyponormal weighted shift ($m \geq 3$), does it follow that W_α is weakly m-hyponormal under a small finite rank perturbation of α?

In this present work we study semi-weakly m-hyponormal weighted shift operators as defined in [6] and give an affirmative answer of the above question for semi-weakly m-hyponormal weighted shift operators introducing a new class of operators call semi m-hyponormal operators.

A bounded linear operator T on a complex Hilbert space H is said to be *semi m-hyponormal* if the operator matrix

$$\begin{pmatrix} [T^*, T] & [T^{*^m}, T] \\ [T^*, T^m] & [T^{*^m}, T^m] \end{pmatrix}$$

on $H \oplus H$ is positive. Obviously semi 2-hyponormality is equivalent to 2-hyponormality. In Section 3, we obtain a characterization for semi m-hyponormal weighted shifts and in Section 4, it has been shown that if W_α is semi m-hyponormal weighted shift, then W_α remains semi-weakly m-hyponormal weighted shift under a small finite rank perturbation of α.

Definition 1.1. [6] An operator T is said to be *semi-weakly m-hyponormal* if $T + sT^m$ is hyponormal for all $s \in \mathbb{C}$.

Semi-weak 2-hyponormality is equivalent to quadratic hyponormality.

2. Preliminaries and notations

Let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a weight sequence in the set of positive real numbers $\mathbb{R}_+$. The weighted shift W_α acting on $\ell^2(\mathbb{N}_0)$, with a canonical orthonormal basis $\{e_i\}_{i=0}^{\infty}$, is defined by $W_\alpha e_i = \alpha_i e_{i+1}$ for all $i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. It is very straightforward that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. A weighted shift W_α is semi-weakly m-hyponormal if and only if $[(W_\alpha + sW_\alpha^m)^*, (W_\alpha + sW_\alpha^m)] \geq 0$ for all $s \in \mathbb{C}$. For $s \in \mathbb{C}$, let

$$D^m(s) := [(W_\alpha + sW_\alpha^m)^*, (W_\alpha + sW_\alpha^m)]$$

$$= (W_\alpha + sW_\alpha^m)^*(W_\alpha + sW_\alpha^m) - (W_\alpha + sW_\alpha^m)(W_\alpha + sW_\alpha^m)^*$$
$$= [W_\alpha^*, W_\alpha] + s[W_\alpha^*, W_\alpha^m] + \bar{s}[W_\alpha^{*^m}, W_\alpha] + |s|^2[W_\alpha^{*^m}, W_\alpha^m].$$

It can be easily shown that

$$[W_\alpha^*, W_\alpha]e_k = (\alpha_k^2 - \alpha_{k-1}^2)e_k \ (\forall\, k \geq 0),$$
$$[W_\alpha^*, W_\alpha^m]e_k = \alpha_k\alpha_{k+1}\dots\alpha_{k+m-2}(\alpha_{k+m-1}^2 - \alpha_{k-1}^2)e_{k+m-1} \ (\forall\, k \geq 0),$$
$$[W_\alpha^{*^m}, W_\alpha]e_k =$$
$$\begin{cases} 0 & \text{if } k = 0, 1, \dots, m-2, \\ \alpha_{k-1}\alpha_{k-2}\dots\alpha_{k-(m-1)}(\alpha_k^2 - \alpha_{k-m}^2)e_{k-(m-1)} & \text{if}\, k \geq m-1, \end{cases}$$
$$[W_\alpha^{*^m}, W_\alpha^m]e_k = (\alpha_k^2\alpha_{k+1}^2\dots\alpha_{k+m-1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2\dots\alpha_{k-m}^2)e_k \ (\forall\, k \geq 0).$$

Let P_n be an orthogonal projection of $\ell^2(\mathbb{N}_0)$ onto $\bigvee_{i=0}^n\{e_i\}$. For any $n \geq m$ and $s \in \mathbb{C}$, let

$$D_n^m(s) := P_n[(W_\alpha + sW_\alpha^m)^*, (W_\alpha + sW_\alpha^m)]P_n.$$

Then

$$D_n^m(s) = \begin{pmatrix} q_{m,0} & 0 & \dots & 0 & z_{m,0} & 0 & & \\ 0 & q_{m,1} & \ddots & \ddots & 0 & z_{m,1} & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & q_{m,m-2} & 0 & \ddots & \ddots & z_{m,n-(m-1)} \\ \bar{z}_{m,0} & 0 & \ddots & 0 & q_{m,m-1} & 0 & \ddots & 0 \\ 0 & \bar{z}_{m,1} & \ddots & \ddots & 0 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & 0 & \ddots & \ddots & q_{m,n-1} & 0 \\ & & 0 & \bar{z}_{m,n-(m-1)} & 0 & \dots & 0 & q_{m,n} \end{pmatrix},$$

where

$$q_{m,k} := u_{m,k} + |s|^2 v_{m,k}, \quad z_{m,k} := \bar{s}\sqrt{w_{m,k}}, \quad u_{m,k} := \alpha_k^2 - \alpha_{k-1}^2,$$
$$v_{m,k} := \alpha_k^2\alpha_{k+1}^2\dots\alpha_{k+m-1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2\dots\alpha_{k-m}^2,$$
$$w_{m,k} := \alpha_k^2\alpha_{k+1}^2\dots\alpha_{k+m-2}^2(\alpha_{k+m-1}^2 - \alpha_{k-1}^2)^2$$

with $\alpha_{-m} = \dots = \alpha_{-2} = \alpha_{-1} = 0$. By definition of semi-weakly m-hyponormal operator, we immediately see that W_α is semi-weakly m-hyponormal if and only if $D_n^m(s) > 0$ for every $n \geq m$ and $s \in \mathbb{C}$. For all $x_0, x_1, \dots, x_n$, $s \in \mathbb{C}$ and $n \geq m$, we define the following:

$$
\begin{aligned}
&F_{m,n}(x_0,\dots,x_n,s)\\
&:=\sum_{k=0}^{n}u_{m,k}|x_k|^2+\sum_{k=0}^{n-(m-1)}s\sqrt{w_{m,k}}x_k\bar{x}_{k+(m-1)}+\sum_{k=0}^{n-(m-1)}\bar{s}\sqrt{w_{m,k}}\bar{x}_kx_{k+m-1}\\
&\quad+\sum_{k=0}^{n}|s|^2v_{m,k}|x_k|^2\\
&=\sum_{k=0}^{n}q_{m,k}|x_k|^2+\sum_{k=0}^{n-(m-1)}\bar{z}_{m,k}x_k\bar{x}_{k+m-1}+\sum_{k=0}^{n-(m-1)}z_{m,k}\bar{x}_kx_{k+m-1},
\end{aligned}
$$

that is, $F_{m,n}(x_0,\dots,x_n,s)=\langle D_n^m(s)X_n,X_n\rangle$, where $X_n=(x_0,x_1,\dots,x_n)^T$. Thus the following result is obvious.

Proposition 2.1. *Let W_α be a weighted shift with positive weight sequence $\alpha=\{\alpha_i\}_{i=0}^{\infty}$. Then the following are equivalent:*

(1) W_α is semi-weakly m-hyponormal;
(2) $\langle D^m(s)X,X\rangle\geq 0$ for any $s\in\mathbb{C}$ and $X\in\ell^2$;
(3) $\langle D_n^m(s)X_n,X_n\rangle\geq 0$ for any $s\in\mathbb{C}$, $X_n=(x_0,x_1,\dots,x_n)^T$ with $x_i\in\mathbb{C}$ and $i=0,1,\dots,n\,(n\geq m)$;
(4) $F_{m,n}(x_0,\dots,x_n,s)\geq 0$ for any $s,x_0,\dots,x_n\in\mathbb{C}$ and $n\geq m$.

For $n\geq m$, we have

$$
\begin{aligned}
&F_{m,n}(x_0,\dots,x_n,s)\\
&=\sum_{k=0}^{n}u_{m,k}|x_k|^2+\sum_{k=0}^{n-(m-1)}s\sqrt{w_{m,k}}x_k\bar{x}_{k+(m-1)}+\sum_{k=0}^{n-(m-1)}\bar{s}\sqrt{w_{m,k}}\bar{x}_kx_{k+m-1}\\
&+\sum_{k=0}^{n}|s|^2v_{m,k}|x_k|^2=\sum_{k=n-(m-2)}^{n}u_{m,k}|x_k|^2+\sum_{k=0}^{m-2}|s|^2v_{m,k}|x_k|^2\\
&+\sum_{k=0}^{n-(m-1)}\left\langle\begin{pmatrix}u_{m,k}&\sqrt{w_{m,k}}\\ \sqrt{w_{m,k}}&v_{m,k+(m-1)}\end{pmatrix}\begin{pmatrix}x_k\\ \bar{s}x_{k+(m-1)}\end{pmatrix},\begin{pmatrix}x_k\\ \bar{s}x_{k+(m-1)}\end{pmatrix}\right\rangle.
\end{aligned}
\tag{2.1}
$$

3. Semi m-hyponormality

Recall that a weighted shift operator W_α is said to be semi m-hyponormal if the operator matrix

$$
\begin{pmatrix}[W_\alpha^*,W_\alpha]&[W_\alpha^{*m},W_\alpha]\\ [W_\alpha^*,W_\alpha^m]&[W_\alpha^{*m},W_\alpha^m]\end{pmatrix}
$$

is positive.

Lemma 3.1. *Let W_α be a hyponormal weighted shift with positive weight sequence $\alpha := \{\alpha_n\}_{n=0}^\infty$. Then W_α is semi m-hyponormal if and only if*

$$\Delta_{m,k} := \begin{pmatrix} u_{m,k} & \sqrt{w_{m,k}} \\ \sqrt{w_{m,k}} & v_{m,k+(m-1)} \end{pmatrix} \geq 0$$

for all $k \geq 0$.

Proof. Consider any two sequences $x = (x_i)$ and $y = (y_i)$ in ℓ^2. Observe that

$$W_\alpha \text{ is semi } m\text{-hyponormal} \Leftrightarrow \begin{pmatrix} [W_\alpha^*, W_\alpha] & [W_\alpha^{*^m}, W_\alpha] \\ [W_\alpha^*, W_\alpha^m] & [W_\alpha^{*^m}, W_\alpha^m] \end{pmatrix} \geq 0,$$

which is equivalent to

$$\begin{aligned} &\langle [W_\alpha^*, W_\alpha]x, x\rangle + \langle [W_\alpha^{*m}, W_\alpha]y, x\rangle + \langle [W_\alpha^*, W_\alpha^m]x, y\rangle + \langle [W_\alpha^{*m}, W_\alpha^m]y, y\rangle \\ =& \sum_{k=0}^\infty \left(u_{m,k}|x_k|^2 + \sqrt{w_{m,k}}(x_k\bar{y}_{k+m-1} + \bar{x}_k y_{k+m-1}) + v_{m,k}|y_k|^2\right) \\ =& \sum_{k=0}^{m-2} v_{m,k}|y_k|^2 + \sum_{k=0}^\infty \left\langle \Delta_{m,k} \begin{pmatrix} x_k \\ y_{k+m-1} \end{pmatrix}, \begin{pmatrix} x_k \\ y_{k+m-1} \end{pmatrix} \right\rangle \geq 0. \end{aligned} \tag{3.1}$$

Since y_k $(0 \leq k \leq m-2)$ is arbitrary, the inequality (3.1) is equivalent to $\Delta_{m,k} \geq 0$ for all $k \geq 0$. $\square$

Lemma 3.1, Proposition 2.1 and equation (2.1) show that semi m-hyponormality implies semi-weak m-hyponormality and Example 3.4 shows that the converse is not true.

Lemma 3.2. [6] *Let W_α be a semi-weakly 3-hyponormal weighted shift with weight sequence $\alpha := \{\alpha_k\}_{k=0}^\infty$. If $\alpha_k = \alpha_{k+1}$ for some $k \geq 1$, then $\alpha_1 = \alpha_2 = \ldots$*

Theorem 3.3. *Let W_α be a semi 3-hyponormal weighted shift with weight sequence $\{\alpha_k\}_{k=0}^\infty$. If $\alpha_k = \alpha_{k-1}$ for some $k \geq 1$, then $\alpha_1 = \alpha_2 = \ldots$*

Proof. By Lemma 3.1, we have W_α is semi 3-hyponormal if and only if

$$\begin{aligned} u_{3,k}\, v_{3,k+2} - w_{3,k} \geq 0 \Leftrightarrow & (\alpha_k^2 - \alpha_{k-1}^2)(\alpha_{k+2}^2\alpha_{k+3}^2\alpha_{k+4}^2 - \alpha_{k+1}^2\alpha_k^2\alpha_{k-1}^2) \\ & \geq \alpha_k^2\alpha_{k+1}^2(\alpha_{k+2}^2 - \alpha_{k-1}^2)^2. \end{aligned}$$

Thus if $\alpha_k = \alpha_{k-1}$, then $\alpha_{k+2} = \alpha_{k-1}$, which implies that $\alpha_{k-1} = \alpha_k = \alpha_{k+1} = \alpha_{k+2}$. Thus by Lemma 3.2, $\alpha_1 = \alpha_2 = \ldots$ $\square$

Example 3.4. By Theorem 3.3, a weighted shift W_α with weight sequence $\alpha : \alpha_0 = \alpha_1 < \alpha_2 < \alpha_3 < \dots$ is not semi 3-hyponormal. But the weighted shift W_α as given in [6] with weight sequence $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{k+1}{k+2}}$ $(k \geq 2)$ is semi-weakly m-hyponormal for $m = 2, 3$.

4. Perturbation

Choose i arbitrarily and fix it. Here $\alpha[i : x]$ denotes the perturbed weight sequence α where the i^{th} weight α_i is replaced by x, for $\alpha_{i-1} < x < \alpha_{i+1}$. Then

$$\Delta'_{m,k} := \begin{pmatrix} u'_{m,k} & \sqrt{w'_{m,k}} \\ \sqrt{w'_{m,k}} & v'_{m,k+m-1} \end{pmatrix},$$

where

$$u'_{m,k} = \begin{cases} u_{m,k}, & \text{for } k < i; \\ x^2 - \alpha_{i-1}^2, & \text{for } k = i; \\ \alpha_{i+1}^2 - x^2, & \text{for } k = i+1; \\ u_{m,k}, & \text{for } k \geq i+2, \end{cases}$$

$$v'_{m,k} = \begin{cases} v_{m,k}, & \text{for } k < i-(m-1); \\ \alpha_k^2 \dots x^2 - \alpha_{k-1}^2 \dots \alpha_{k-m}^2, & \text{for } k = i-(m-1); \\ \alpha_k^2 \dots x^2 \alpha_{k+m-1}^2 - \alpha_{k-1}^2 \dots \alpha_{k-m}^2, & \text{for } k = i-(m-2); \\ \vdots & \\ x^2 \dots \alpha_{k+m-1}^2 - \alpha_{k-1}^2 \dots \alpha_{k-m}^2, & \text{for } k = i; \\ \alpha_k^2 \dots \alpha_{k+m-1}^2 - x^2 \dots \alpha_{k-m}^2, & \text{for } k = i+1; \\ \vdots & \\ \alpha_k^2 \dots \alpha_{k+m-1}^2 - \alpha_{k-1}^2 \dots x^2, & \text{for } k = i+m; \\ v_{m,k}, & \text{for } k \geq i+(m+1), \end{cases}$$

$$w'_{m,k} = \begin{cases} w_{m,k}, & \text{for } k < i-(m-1); \\ \alpha_{i-(m-1)}^2 \alpha_{i-(m-2)}^2 \dots \alpha_{i-1}^2 (x^2 - \alpha_{i-m}^2)^2, & \text{for } k = i-(m-1); \\ \alpha_{i-(m-2)}^2 \alpha_{i-(m-3)}^2 \dots x^2 (\alpha_{i+1}^2 - \alpha_{i-(m-2)}^2)^2, & \text{for } k = i-(m-2); \\ \vdots & \\ x^2 \alpha_{i+1}^2 \dots \alpha_{i+(m-2)}^2 (\alpha_{i+(m-1)}^2 - \alpha_{i-1}^2)^2, & \text{for } k = i; \\ \alpha_{i+1}^2 \alpha_{i+2}^2 \dots \alpha_{i+(m-1)}^2 (\alpha_{i+m}^2 - x^2)^2, & \text{for } k = i+1; \\ w_{m,k}, & \text{for } k \geq i+2. \end{cases}$$

Clearly $\Delta'_{m,k} = \Delta_{m,k}$ for all k except $k = i-(2m-2), i-(2m-3), \dots, i-1, i, i+1$.

Theorem 4.1. *Let W_α be a semi m-hyponormal weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$. Let the 0^{th} weight α_0 be slightly perturbed to say x, and let $W_{\alpha[0:x]}$ denote the perturbed shift with weight sequence $\alpha[0:x] := \{\alpha'_n\}_{n=0}^{\infty}$ given by $\alpha'_0 = x,\ \alpha'_n = \alpha_n$ for $n > 0$. Then we have:*

(1) If $\det \Delta_{m,1} > 0$ then there exists $\varepsilon > 0$ such that $W_{\alpha[0:x]}$ is semi m-hyponormal for all $x \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$.

(2) If $\det \Delta_{m,1} = 0$ then there exists $\varepsilon > 0$ such that $W_{\alpha[0:x]}$ is semi m-hyponormal for all $x \in (\alpha_0 - \varepsilon, \alpha_0)$ but $W_{\alpha[0:x]}$ is not semi m-hyponormal for $x \in (\alpha_0, \alpha_0 + \varepsilon)$.

Proof. By Lemma 3.1, $W_{\alpha[0:x]}$ is semi m-hyponormal if and only if $\Delta'_{m,k} \geq 0$ for all $k \geq 0$. Referring to the notations introduced above, we have $\Delta'_{m,k} = \Delta_{m,k}$ for all k except $k = 0, 1$. Thus we only need to check the positivity of $\Delta'_{m,0}$ and $\Delta'_{m,1}$.
Positivity of $\Delta'_{m,0}$:

$$\Delta'_{m,0} := \begin{pmatrix} u'_{m,0} & \sqrt{w'_{m,0}} \\ \sqrt{w'_{m,0}} & v'_{m,\,m-1} \end{pmatrix},$$

where $u'_{m,\,0} = x^2,\ w'_{m,\,0} = x^2 \ldots \alpha_{m-2}^2\, \alpha_{m-1}^4$ and $v'_{m,\,m-1} = \alpha_{m-1}^2 \alpha_m^2 \ldots \alpha_{2m-2}^2$. Hence

$$\det \Delta'_{m,0} = x^2 \alpha_{m-1}^2 \big\{ (\alpha_m^2 \ldots \alpha_{2m-2}^2 - \alpha_1^2 \ldots \alpha_{m-2}^2\, \alpha_{m-1}^2) \big\} \geq 0,$$

for all $0 < x < \alpha_1$ and so $\Delta'_{m,0} \geq 0$.
Positivity of $\Delta'_{m,1}$:

$$\Delta'_{m,1} := \begin{pmatrix} u'_{m,1} & \sqrt{w'_{m,1}} \\ \sqrt{w'_{m,1}} & v'_{m,\,m} \end{pmatrix},$$

where $u'_{m,\,1} = \alpha_1^2 - x^2,\ w'_{m,\,1} = \alpha_1^2\, \alpha_2^2 \ldots \alpha_{m-1}^2 (\alpha_m^2 - x^2)^2$ and $v'_{m,\,m} = \alpha_m^2 \ldots \alpha_{2m-1}^2 - \alpha_{m-1}^2 \ldots x^2$. Let $f(x) = \det \Delta'_{m,1}$. Then

$$\begin{aligned} f(x) = {} & x^2 \left[\alpha_1^2\, \alpha_2^2 \ldots \alpha_{m-1}^2 (\alpha_m^2 - \alpha_1^2) - \alpha_m^2 \big(\alpha_{m+1}^2 \ldots \alpha_{2m-1}^2 - \alpha_1^2\, \alpha_2^2 \ldots \alpha_{m-1}^2 \big) \right] \\ & + \big(\alpha_1^2\, \alpha_m^2 \ldots \alpha_{2m-1}^2 - \alpha_1^2\, \alpha_2^2 \ldots \alpha_{m-1}^2 \alpha_m^4 \big) . \end{aligned}$$

For $x = \alpha_0,\ f(\alpha_0) = \det \Delta_{m,\,1} \geq 0$. Now if $\det \Delta_{m,\,1} > 0$, then $f(\alpha_0) > 0$, so by continuity of f, there exists $\epsilon > 0$ such that $f(x) > 0$ for all $x \in (\alpha_0 - \epsilon, \alpha_0 + \epsilon)$. But suppose $\det \Delta_{m,\,1} = 0$. Then $f(\alpha_0) = 0$, which implies

$$\left[\alpha_1^2\, \alpha_2^2 \ldots \alpha_{m-1}^2 (\alpha_m^2 - \alpha_1^2) - \alpha_m^2 \big(\alpha_{m+1}^2 \ldots \alpha_{2m-1}^2 - \alpha_1^2\, \alpha_2^2 \ldots \alpha_{m-1}^2 \big) \right]$$

$$= -\frac{1}{\alpha_0^2}\,\alpha_1^2\alpha_m^2\left(\alpha_{m+1}^2\dots\alpha_{2m-1}^2-\alpha_2^2\dots\alpha_{m-1}^2\alpha_m^2\right). \tag{4.1}$$

Thus, by using (4.1) we have

$$\begin{aligned}\frac{d}{dx}f(x) &= 2x\left[\alpha_1^2\,\alpha_2^2\dots\alpha_{m-1}^2(\alpha_m^2-\alpha_1^2)-\alpha_m^2\big(\alpha_{m+1}^2\dots\alpha_{2m-1}^2-\alpha_1^2\,\alpha_2^2\dots\alpha_{m-1}^2\big)\right]\\ &= -2x\frac{1}{\alpha_0^2}\,\alpha_1^2\alpha_m^2\left(\alpha_{m+1}^2\dots\alpha_{2m-1}^2-\alpha_2^2\dots\alpha_{m-1}^2\alpha_m^2\right)<0,\end{aligned}$$

for all $x>0$. This shows that the continuous function f is decreasing and as $f(\alpha_0)=0$, we conclude that there exists $\epsilon>0$ such that $f(x)>0$ for $x\in(\alpha_0-\epsilon,\alpha_0)$ and $f(x)<0$ for $x\in(\alpha_0,\alpha_0+\epsilon)$. □

Thus we have shown that under certain conditions a perturbed semi m-hyponormal weighted shift may not again be semi m-hyponormal. But in the main result Theorem 4.3, we have shown that a semi m-hyponormal weighted shift remains semi-weakly m-hyponormal under small non-zero finite rank perturbations.

Lemma 4.2. [6] *Let $m\geq 2$ be a fixed positive integer. Then for $n\in\mathbb{N}$ with $n\geq m-1, n=(m-1)\ell+j$, $D_n^m(s)$ is unitarily equivalent to $[\bigoplus_{i=0}^{j}D_{j,i,1}^m(\ell)]\bigoplus[\bigoplus_{i=j+1}^{m-2}D_{j,i,2}^m(\ell)]$, where*

$$D_{j,i,1}^m(\ell):=\begin{pmatrix} q_{m,\,j-i} & z_{m,\,j-i} & 0 & & \\ \bar{z}_{m,\,j-i} & q_{m,\,(m-1)+j-i} & \ddots & \ddots & \\ 0 & \bar{z}_{m,\,(m-1)+j-i} & \ddots & \ddots & 0 \\ & \ddots & \ddots & q_{m,\,(\ell-1)\,(m-1)+j-i} & z_{m,\,(\ell-1)\,(m-1)+j-i} \\ & & 0 & \bar{z}_{m,\,(\ell-1)\,(m-1)+j-i} & q_{m,\,\ell\,(m-1)+j-i}\end{pmatrix}$$

for $0\leq i\leq j$, and $D_{j,i,2}^m(\ell)$

$$:=\begin{pmatrix} q_{m,\,(m-1)+j-i} & z_{m,\,(m-1)+j-i} & 0 & & \\ \bar{z}_{m,\,(m-1)+j-i} & q_{m,\,2(m-1)+j-i} & \ddots & \ddots & \\ 0 & \bar{z}_{m,\,2(m-1)+j-i} & \ddots & \ddots & 0 \\ & \ddots & \ddots & q_{m,\,(\ell-1)\,(m-1)+j-i} & z_{m,\,(\ell-1)\,(m-1)+j-i} \\ & & 0 & \bar{z}_{m,\,(\ell-1)\,(m-1)+j-i} & q_{m,\,\ell\,(m-1)+j-i}\end{pmatrix}$$

for $j+1\leq i\leq m-2$.

A weighted shift W_α is semi-weakly m-hyponormal if and only if for all $j\,(0 < j < m-2)$ and $\ell\,(\ell \geq 1), D^m_{j,i,1}(\ell) \geq 0\,(0 \leq i \leq j)$ and $D^m_{j,i,2}(\ell) \geq 0\,(j+1 \leq i \leq m-2)$.

Theorem 4.3. *(Rank-1 perturbation) Let $\alpha = \{\alpha_i\}_{i=0}^\infty$ be a strictly increasing positive weight sequence and W_α be a semi m-hyponormal weighted shift. For $\alpha_{i-1} < x < \alpha_{i+1}$, let $\alpha[i:x]$ denote the weight sequence α with the i^{th} weight α_i replaced by x. Then there exists $\epsilon > 0$ such that $W_{\alpha[i:x]}$ is semi-weakly m-hyponormal for $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$.*

Proof. Let $\Delta'_{m,k}, u'_{m,k}, v'_{m,k}, w'_{m,k}$ be defined (as in section 4) with respect to $W_{\alpha[i:x]}$. For $x_0, \ldots, x_n, s \in \mathbb{C}$ and $n \geq m$, define

$$\begin{aligned} F'_{m,n}(x_0,\ldots,x_n,s) &:= \sum_{k=n-(m-2)}^{n} u'_{m,k}|x_k|^2 + \sum_{k=0}^{m-2} |s|^2 v'_{m,k}|x_k|^2 \\ &+ \sum_{k=0}^{n-(m-1)} \left\langle \begin{pmatrix} u'_{m,k} & \sqrt{w'_{m,k}} \\ \sqrt{w'_{m,k}} & v'_{m,k+(m-1)} \end{pmatrix} \begin{pmatrix} x_k \\ \bar{s}x_{k+(m-1)} \end{pmatrix}, \begin{pmatrix} x_k \\ \bar{s}x_{k+(m-1)} \end{pmatrix} \right\rangle. \end{aligned} \tag{4.2}$$

Thus $W_{\alpha[i:x]}$ is semi-weakly m-hyponormal if $F'_{m,n}(x_0,\ldots,x_n,s) \geq 0$ for all $x_0,\ldots,x_n, s \in \mathbb{C}$ and $n \geq m$. For $s = 0$, by equation (4.2), $F'_{m,n}(x_0,\ldots,x_n,s) \geq 0$. Assume $s \neq 0$. Then equation (4.2) is equivalent to

$$F'_{m,n}(x_0,\ldots,x_n,s) = \sum_{k=n-(m-2)}^{n} u'_{m,k}|x_k|^2 + \sum_{i=0}^{j} \langle A'^m_{j,i,1}(\ell)X, X\rangle + \sum_{i=j+1}^{m-2} \langle A'^m_{j,i,2}(\ell)X, X\rangle,$$

where $X = (x_r, \bar{s}x_{r+(m-1)}, \bar{s}^2 x_{r+2(m-1)} \ldots, \bar{s}^\ell x_{r+\ell(m-1)})^T$ $(r = j-i)$ and $A'^m_{j,i,1}(\ell)$ is the matrix

$$\begin{pmatrix} |s|^2 {v'}_{m,j-i} + {u'}_{m,j-i} & \sqrt{{w'}_{m,j-i}} & \cdots & 0 \\ \sqrt{{w'}_{m,j-i}} & {v'}_{m,(m-1)+j-i} + \frac{{u'}_{m,(m-1)+j-i}}{|s|^2} & \cdots & 0 \\ 0 & \frac{\sqrt{{w'}_{m,(m-1)+j-i}}}{|s|^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\sqrt{{w'}_{m,(\ell-1)(m-1)+j-i}}}{|s|^{2(\ell-1)}} \\ 0 & 0 & \cdots & \frac{{v'}_{m,\ell(m-1)+j-i}}{|s|^{2(\ell-1)}} \end{pmatrix}$$

for $0 \leq i \leq j$, and $A'^m_{j,i,2}(\ell)$ is the matrix

$$\begin{pmatrix} |s|^2 {v'}_{m,(m-1)+j-i} + {u'}_{m,(m-1)+j-i} & \sqrt{{w'}_{m,(m-1)+j-i}} & \cdots & 0 \\ \sqrt{{w'}_{m,(m-1)+j-i}} & {v'}_{m,2(m-1)+j-i} + \frac{{u'}_{m,2(m-1)+j-i}}{|s|^2} & \cdots & 0 \\ 0 & \frac{\sqrt{{w'}_{m,2(m-1)+j-i}}}{|s|^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \\ 0 & 0 & \cdots & \frac{{v'}_{m,\ell(m-1)+j-i}}{|s|^{2(\ell-1)}} \end{pmatrix}$$

for $j+1 \le i \le m-2$. For $q = 1, 2$ and all relevant (i, j), the matrix $A'^m_{j,i,q}(\ell)$ has the form

$$\begin{pmatrix} |s|^2 {v'}_{m,r} + {u'}_{m,r} & \sqrt{{w'}_{m,r}} & \cdots & 0 \\ \sqrt{{w'}_{m,r}} & {v'}_{m,r+(m-1)} + \frac{{u'}_{m,r+(m-1)}}{|s|^2} & \cdots & 0 \\ 0 & \frac{\sqrt{{w'}_{m,r+(m-1)}}}{|s|^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\sqrt{{w'}_{m,r+(\ell-1)(m-1)}}}{|s|^{2(\ell-1)}} \\ 0 & 0 & \cdots & \frac{{v'}_{m,r+\ell(m-1)}}{|s|^{2(\ell-1)}} \end{pmatrix}$$

for some r satisfying $0 \le r \le m-2$. This is obvious for $A'^m_{j,i,1}(\ell)$ with $r = j - i$ and $(0 \le r \le j)$ and is an easy computation for the $A'^m_{j,i,2}(\ell)$ with $(j+1 \le r \le m-2)$. Thus to show W_α is semi-weakly m- hyponormal, it is sufficient to verify the positivity of the matrix $A'^m_{j,i,q}(\ell)$. Define $B'^m_{j,i,q}(\ell)$ as

$$\begin{pmatrix} |s|^2 {v'}_{m,r} + {u'}_{m,r} & \sqrt{{w'}_{m,r}} & \cdots & 0 \\ \sqrt{{w'}_{m,r}} & {v'}_{m,r+(m-1)} + \frac{{u'}_{m,r+(m-1)}}{|s|^2} & \cdots & 0 \\ 0 & \frac{\sqrt{{w'}_{m,r+(m-1)}}}{|s|^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{{v'}_{m,r+(\ell-1)(m-1)}}{|s|^{2(\ell-2)}} + \frac{{u'}_{m,r+(\ell-1)(m-1)}}{|s|^{2(\ell-1)}} \end{pmatrix}$$

and $\psi_1(x) := \det A'^m_{j,i,q}(1) = |s|^2 v'_{m,r} v'_{m,r+(m-1)} + (u'_{m,r} v'_{m,r+(m-1)} - w'_{m,r})$, $\psi_\ell(x) := \det A'^m_{j,i,q}(\ell)$, $\varphi_1(x) := |s|^2 {v'}_{m,r} + {u'}_{m,r}$ and $\varphi_\ell(x) := \det B'^m_{j,i,q}(\ell)$. Therefore,

$\varphi_{\ell+1}(x) = \det B'^m_{j,i,q}(\ell+1) = \left(\frac{v'_{m,r+\ell(m-1)}}{|s|^{2(\ell-1)}} + \frac{u'_{m,r+\ell(m-1)}}{|s|^{2\ell}} \right) \det B'^m_{j,i,q}(\ell) + Q$,

where Q is the remaining part of the determinant and

$\psi_\ell(x) = \frac{v'_{m,r+\ell(m-1)}}{|s|^{2(\ell-1)}} \det B'^m_{j,i,q}(\ell) + Q$. Thus $\varphi_{\ell+1}(x) = \psi_\ell(x) + \frac{u'_{m,r+\ell(m-1)}}{|s|^{2\ell}} \varphi_\ell(x)$ and $\psi_{\ell+1}(x) = \det A'^m_{j,i,q}(\ell+1)$

$$
\begin{aligned}
&= \frac{v'_{m,r+(\ell+1)(m-1)}}{|s|^{2\ell}} \det B'^{m}_{j,i,q}(\ell+1) - \frac{w'_{m,r+\ell(m-1)}}{|s|^{4\ell}} \det B'^{m}_{j,i,q}(\ell) \\
&= \frac{v'_{m,r+(\ell+1)(m-1)}}{|s|^{2\ell}} \varphi_{\ell+1}(x) - \frac{w'_{m,r+\ell(m-1)}}{|s|^{4\ell}} \varphi_{\ell}(x) \\
&= \frac{v'_{m,r+(\ell+1)(m-1)}}{|s|^{2\ell}} \big(\psi_\ell(x) + \frac{u'_{m,r+\ell(m-1)}}{|s|^{2\ell}} \varphi_\ell(x)\big) - \frac{w'_{m,r+\ell(m-1)}}{|s|^{4\ell}} \varphi_{\ell}(x) \\
&= \frac{v'_{m,r+(\ell+1)(m-1)}}{|s|^{2\ell}} \psi_\ell(x) + \frac{\varphi_\ell(x)}{|s|^{4\ell}} \big(u'_{m,r+\ell(m-1)} v'_{m,r+(\ell+1)(m-1)} - w'_{m,r+\ell(m-1)}\big) \\
&= \frac{v'_{m,r+(\ell+1)(m-1)}}{|s|^{2\ell}} \psi_\ell(x) + \frac{\varphi_\ell(x)}{|s|^{4\ell}} \det \Delta'_{m,r+\ell(m-1)}. \qquad (4.3)
\end{aligned}
$$

Clearly $\varphi_1(x) > 0$ for all $\alpha_{i-1} < x < \alpha_{i+1}$. Now, $\psi_1(x) = |s|^2 v'_{m,r} v'_{m,r+(m-1)} +$ $\det \Delta'_{m,r}$. At $x = \alpha_i$, $\psi_1(x) = |s|^2 v_{m,r} v_{m,r+(m-1)} + \det \Delta_{m,r} > 0$ as W_α is semi m-hyponormal. Now, $\varphi_1(\alpha_i), \psi_1(\alpha_i) > 0$ gives $\varphi_2(\alpha_i) > 0$. Also

$$\psi_2(\alpha_i) = \frac{v_{m,r+2(m-1)}}{|s|^2} \psi_1(\alpha_i) + \frac{\varphi_1(\alpha_i)}{|s|^4} \det \Delta_{m,r+(m-1)} > 0.$$

So by continuity of ψ_2, there exists $\epsilon > 0$ such that $\psi_2(x) > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. Repeating the same argument we can conclude that there exists $\epsilon > 0$ such that $\varphi_\ell(x), \psi_\ell(x) > 0$ for all $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. In other words, $A'^{m}_{j,i,q}(\ell) > 0$ for all $\ell \geq 0$ and $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$. ☐

Example 4.4. Consider the weighted shift W_α with weight sequence $\alpha = \{\alpha_k\}_{k=0}^{\infty}$, where $\alpha_k = \sqrt{\frac{k+1}{k+2}}$. Then W_α is semi 3-hyponormal because $\det \Delta_{3,k}$ $= \frac{18(2k+5)}{(k^2+5k+4)^2(k^2+5k+6)} > 0$ for all $k \geq 0$. Thus by Theorem 4.3, for $i = 0, 1, 2, \ldots$ there exists $\epsilon > 0$ for which $W_{\alpha[i:x]}$ is semi-weakly 3-hyponormal for each $x \in (\alpha_i - \epsilon, \alpha_i + \epsilon)$.

Conclusion: Theorem 4.3 can be generalized to finite rank perturbation of the weight sequence by following the similar process of simplification.

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SOLUTION OF SINGLE PARAMETER BRING QUINTIC EQUATION

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Abstract. In this paper, we propose a new method to obtain a solution to a single-parameter Bring quintic equation of the form, $x^5 + x = a$, where a is real. The method transforms the given quintic equation to an infinite but convergent series expression in (x/a), which is further transformed to a quartic equation in a novel fashion. The coefficients of the quartic equation so obtained are some kind of infinite series expressions in a^{-4}, which are termed as *ultraradicals*. The quartic equation is then solved and its one real solution is picked; further using this, the real solution of quintic equation, $x^5 + x = a$, is extracted. The ultraradicals used in this method converge for $|a| > 1$; hence the method can be used when $|a| > 1$.

1. Introduction

Many real world applications encounter quintic equations. For example, the three-body motion of celestial objects governed by Keppler laws requires solution to a quintic equation to determine the position of the objects [1]. In hydraulic engineering, the height of water flowing in an open rectangular channel is obtained by solving a quintic equation [2]. In structural mechanics, the study of nonlinear vibration of beams involves quintic equations [3]. Since the general quintic equations do not have closed form solutions (similar to that exist for cubic and quartic equations), one has to resort to either numerical methods or use special functions (called *ultraradicals*) to obtain the solutions. A brief discussion on the historical account of efforts put in by several mathematicians for solving quintic equations is given below.

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The cubic and the quartic equations were solved by Cardan and Ferrari respectively in the sixteenth century; and the intense struggle by several mathematicians to solve the quintic equation, using the techniques similar to that adopted for cubic and quartic equations met with no success. In 1683, the German mathematician Ehrenfried Walther von Tschirnhaus introduced a polynomial transformation bearing his name, which could eliminate intermediate terms in an n-th degree polynomial equation. Using the transformation, Tschirnhaus showed that the cubic equation can be solved by removing the two intermediate terms. Further, the quartic equation was solved by removing the two odd-power terms [4]. This result encouraged the mathematicians to explore the ways for reducing a polynomial equation to simpler forms, hoping that it might lead to the solutions of the quintic and the higher degree equations.

In 1786, the Swedish mathematician E. S. Bring showed that one can transform the general quintic equation to the form $x^5 + Ax + B = 0$. It appears that the papers containing the works of Bring got lost in the archives of University of Lund. Probably unaware of Bring's work, the French mathematician G. B. Jerrard (1852) also reduced the general quintic equation to the form $x^5 + Ax + B = 0$ [4, 5]; for detailed description of getting this form using Tschirnhaus transformation, see [6, 7].

However, all these efforts of reducing the polynomial equations to simpler forms could not achieve the desired goal of solving the quintic and the higher degree equations. The issue of unsolvability of these equations was eventually resolved by Abel (1824) and Galois (1832), who conclusively proved that the general polynomial equations of degree five and above cannot be solved by the methods similar to that were adopted for solving the cubic and the quartic equations [4]. This means that these equations cannot be solved by performing finite number of elementary arithmetic operations (addition, subtraction, multiplication, and division) along with the radical (n-th root) operation, which were enough to solve the quadratic, the cubic, or the quartic equations. In other words, quintic and the higher-degree equations cannot be solved using radicals alone.

The simplest form of quintic equation that is unsolvable in radicals is Bring-Jerrard quintic equation, $x^5 + Ax + B = 0$. This doesn't mean that there are no solutions to the quintic equations; the solutions of quintics can still be obtained (without violating the Galois theory) by the use of more complicated functions than the radicals. Notice that the radical (n-th root, $n = 2, 3, 4$) operation can be viewed as a transcendental function involving a logarithmic or trigonometric function in a parameter [8]. The complicated functions that are useful for solving the quintic and the higher-degree equations are now termed as *ultraradicals* [8]. Elliptic modular functions are one form of ultraradicals used by Hermite (1858) for solving the single-parameter Bring quintic equation, $x^5 - x - a = 0$ [9, 10].

Also we notice from the literature that the Bring quintic equation of the form, $x^5 + x = a$ (with a real), is solved using an ultraradical called Bring radical, $BR(a)$, which is an infinite series function in a, as shown below [7, 11].

$$x = BR(a) = a - a^5 + 5a^9 - 35a^{13} + 285a^{17} - 2530a^{21} + \dots$$

The above series is convergent for values of $|a| < 4/(\,5 \cdot \sqrt[4]{5}\,)$, see [11].

In this paper, we present a method for extracting a real solution of the quintic equation, $x^5 + x = a$ (with a real), by transforming it to a quartic equation, whose coefficients are ultraradicals. The ultraradicals in our case happen to be some kind of infinite series functions in a^{-4}, as one can see later in this paper. These series functions are convergent for $|a| > 1$; so the proposed method can be used when $|a| > 1$.

2. The proposed method

Consider the following single-parameter Bring quintic equation,

$$x^5 + x = a, \tag{2.1}$$

where a is real. Since any polynomial of odd-degree in x (with real coefficients) crosses the x-axis at least once, the quintic equation (2.1) has at least one real solution. We plan to extract this solution algebraically. For this purpose, when (2.1) is rearranged as, $(x/a) = 1/(1 + x^4)$, it is easier to notice that the real root (x) of quintic equation (2.1) satisfies the inequality, $0 < (x/a) < 1$, since x^4 is always a positive number. Equipped with this information, (2.1) is further rearranged as,

$$(x/a)^5 = a^{-4}[1 - (x/a)]. \tag{2.2}$$

Taking the fifth root of (2.2) yields,

$$(x/a) = a^{-4/5}[1 - (x/a)]^{1/5}. \tag{2.3}$$

Expanding the right-hand-side (RHS) of (2.3) using binomial theorem results in,

$$(x/a) = a^{-4/5}\left[1 + \sum_{k=1}^{\infty} \frac{n(n-1)(n-2)\ ...\ k \text{ terms}}{k!}\left(\frac{-x}{a}\right)^k\right], \tag{2.4}$$

where $n = 1/5$. Expressing the infinite series in (2.4) explicitly and rearranging the terms yields the following expression,

$$\begin{aligned} a^{4/5}(x/a) = 1 &- c_1(x/a) - c_2(x/a)^2 - c_3(x/a)^3 - c_4(x/a)^4 - c_5(x/a)^5 \\ &- c_6(x/a)^6 - c_7(x/a)^7 - c_8(x/a)^8 - c_9(x/a)^9 - c_{10}(x/a)^{10} - \ ..., \end{aligned} \tag{2.5}$$

where 'c_k's are evaluated using the recurrence formula,

$$c_{k+1} = [(5k-1)/5(k+1)]c_k,$$

and are listed in Table 1. Since $0 < (x/a) < 1$ and $0 < c_{k+1} < c_k < 1$, it

TABLE 1. Values of c_k for $k = 1$ to 36

k	c_k	k	c_k	k	c_k
1	0.2	13	0.007986216075264	25	0.003627166639633
2	0.08	14	0.007301683268813	26	0.003459758948573
3	0.048	15	0.006717548607308	27	0.003305991884192
4	0.0336	16	0.00621373246176	28	0.003164306517727
5	0.025536	17	0.005775116052694	29	0.003033369696303
6	0.0204288	18	0.005390108315848	30	0.002912034908451
7	0.01692672	19	0.005049680422216	31	0.002799310976511
8	0.014387712	20	0.004746699596883	32	0.002694336814892
9	0.0124693504	21	0.004475459619918	33	0.002596360930714
10	0.010973028352	22	0.00423134364065	34	0.002504724662571
11	0.0097759707136	23	0.004010577885485	35	0.002418848388426
12	0.00879837364224	24	0.003810048991211	36	0.002338220108812

implies that the terms in the infinite series in (2.5) satisfy the inequality, $c_{k+1}(x/a)^{k+1} < c_k(x/a)^k$; hence we conclude that the series is a convergent infinite series. Rearranging (2.5) yields,

$$\begin{aligned}1 - (c_1 + a^{4/5})(x/a) - c_2(x/a)^2 - c_3(x/a)^3 - c_4(x/a)^4 - c_5(x/a)^5 \\ -c_6(x/a)^6 - c_7(x/a)^7 - c_8(x/a)^8 - c_9(x/a)^9 - ... = 0, \quad (2.6)\end{aligned}$$

Now we use (2.2) repeatedly to eliminate $(x/a)^5$ and higher power terms from (2.6), which results in a quartic equation in (x/a) as below.

$$K_4(x/a)^4 + K_3(x/a)^3 + K_2(x/a)^2 + K_1(x/a) + K_0 = 0, \qquad (2.7)$$

where K_0, K_1, K_2, K_3, and K_4 are given by:

$$\begin{aligned}K_0 &= 1 - (c_5/a^4) + [(c_9 - c_{10})/a^8] - [(c_{13} - 2c_{14} + c_{15})/a^{12}] + \ ... \\ K_1 &= -a^{4/5} - c_1 + [(c_5 - c_6)/a^4] - [(c_9 - 2c_{10} + c_{11})/a^8] \\ &\quad +[(c_{13} - 3c_{14} + 3c_{15} - c_{16})/a^{12}] - \ ... \\ K_2 &= -c_2 + [(c_6 - c_7)/a^4] - [(c_{10} - 2c_{11} + c_{12})/a^8] \\ &\quad +[(c_{14} - 3c_{15} + 3c_{16} - c_{17})/a^{12}] - \ ... \\ K_3 &= -c_3 + [(c_7 - c_8)/a^4] - [(c_{11} - 2c_{12} + c_{13})/a^8] \\ &\quad +[(c_{15} - 3c_{16} + 3c_{17} - c_{18})/a^{12}] - \ ... \\ K_4 &= -c_4 + [(c_8 - c_9)/a^4] - [(c_{12} - 2c_{13} + c_{14})/a^8] \\ &\quad +[(c_{16} - 3c_{17} + 3c_{18} - c_{19}/a^{12}] + \ ... \quad (2.8)\end{aligned}$$

Notice that the coefficients, K_0, K_1, K_2, K_3, and K_4, given in (2.8) are infinite series expressions in a^{-4}, and are termed as ultraradicals in a, similar to that defined in [8]. The convergence of these functions will be discussed in the next section. Normalizing the quartic equation (2.7) by dividing it throughout by K_4 yields,

$$\begin{aligned}(x/a)^4 + (K_3/K_4)(x/a)^3 + (K_2/K_4)(x/a)^2 \\ +(K_1/K_4)(x/a) + (K_0/K_4) = 0. \qquad (2.9)\end{aligned}$$

Solving (2.9) by traditional methods [12, 13] yields four solutions of (x/a). Since we already know that a real solution of the given quintic equation (2.1) satisfies the inequality, $0 < (x/a) < 1$, we now pick a solution of (2.9), which is real, positive, and less than unity, say x_1/a; from this the desired solution of (2.1) is obtained as x_1.

3. Salient features of ultraradicals

In order to study the salient features of the ultraradicals defined above, we first express them in general forms as below.

$$
\begin{aligned}
K_0 &= 1 + \sum_{m=1}^{\infty}\left[\frac{(-1)^m}{a^{4m}}\sum_{n=0}^{m-1}(-1)^n C_n^{m-1} c_{4m+n+1}\right], \\
K_1 &= -a^{4/5} + \sum_{m=1}^{\infty}\left[\frac{(-1)^m}{a^{4(m-1)}}\sum_{n=0}^{m-1}(-1)^n C_n^{m-1} c_{4m+n-3}\right], \\
K_2 &= \sum_{m=1}^{\infty}\left[\frac{(-1)^m}{a^{4(m-1)}}\sum_{n=0}^{m-1}(-1)^n C_n^{m-1} c_{4m+n-2}\right], \\
K_3 &= \sum_{m=1}^{\infty}\left[\frac{(-1)^m}{a^{4(m-1)}}\sum_{n=0}^{m-1}(-1)^n C_n^{m-1} c_{4m+n-1}\right], \\
K_4 &= \sum_{m=1}^{\infty}\left[\frac{(-1)^m}{a^{4(m-1)}}\sum_{n=0}^{m-1}(-1)^n C_n^{m-1} c_{4m+n}\right].
\end{aligned}
\tag{3.1}
$$

Further, making use of numerical values given in Table 1, the expressions in (3.1) are written as:

$$
\begin{aligned}
K_0 &= 1 - (0.025536)a^{-4} + (0.001496322...)a^{-8} \\
&\quad -(0.000100398...)a^{-12} + (0.000007132...)a^{-16} - \ldots, \\
K_1 &= -a^{4/5} - 0.2\left[1 - (0.025536)a^{-4} + (0.001496322...)a^{-8}\right. \\
&\quad \left. -(0.000100398...)a^{-12} + (0.000007132...)a^{-16} - \ldots\right], \\
K_2 &= 0.92 - \left[1 - (0.00350208)a^{-4} + (0.0002194606...)a^{-8}\right. \\
&\quad \left. -(0.0000151188...)a^{-12} + (0.0000010894...)a^{-16} - \ldots\right], \\
K_3 &= 0.952 - \left[1 - (0.002539008)a^{-4} + (0.0001654395...)a^{-8}\right. \\
&\quad \left. -(0.0000115911...)a^{-12} + (0.0000008431...)a^{-16} - \ldots\right], \\
K_4 &= 0.9664 - \left[1 - (0.0019183616...)a^{-4} + (0.0001276248...)a^{-8}\right. \\
&\quad \left. -(0.0000090288...)a^{-12} + (0.0000006604...)a^{-16} - \ldots\right].
\end{aligned}
\tag{3.2}
$$

It is essential to know for which values of a the series expressions in (3.2) are convergent, so that the proposed method becomes relevant in that range. Observing the expressions given in (3.2), we notice that $K_1 = -a^{4/5} - 0.2K_0$. Further, a comparison of infinite series in K_2 with that of K_0 reveals that the magnitudes of coefficients of a^{-4m} in K_2 are smaller than that in K_0

$(m = 1, 2, 3, \ldots)$; hence, if K_0 is convergent for some value of a, K_2 also will be convergent for that a.

Similarly the magnitudes of coefficients of a^{-4m} in K_3 and K_4 are also smaller than that in K_0. Hence K_3 and K_4 will also be convergent for that value of a, for which K_0 is convergent. Therefore, it is sufficient if we determine the convergence of K_0, and this can be done more easily with experimental techniques than the theoretical ones. For this purpose, we determine first few terms (T_m) in K_0. The first forty-one terms in K_0 are determined using the expression in (3.1) and listed in Table 2, for $|a| = 1$. The partial sums of the (first) eleven terms (S_{11}), twenty-one terms (S_{21}), thirty-one terms (S_{31}), and the forty-one terms (S_{41}) for various values of $|a|$ are determined and listed in Table 3.

TABLE 2. First forty-one terms in K_0 for $|a| = 1$.

m	T_m	m	T_m	m	T_m
0	1	14	-4.18719E-16	28	-5.91874E-13
1	-0.025536	15	2.03407E-15	29	3.10788E-12
2	0.001496322	16	-8.04998E-15	30	4.96923E-12
3	-0.000100398	17	-3.5667E-15	31	-1.00878E-11
4	7.13277E-06	18	3.38033E-15	32	3.2447E-12
5	-5.23045E-07	19	-3.77267E-14	33	-1.26564E-10
6	3.91345E-08	20	3.64555E-14	34	-1.60997E-11
7	-2.96913E-09	21	9.31319E-14	35	1.56419E-10
8	2.27581E-10	22	1.88169E-14	36	-1.14683E-10
9	-1.75807E-11	23	4.60115E-14	37	3.06899E-10
10	1.36658E-12	24	8.33396E-14	38	9.61151E-10
11	-1.06755E-13	25	-6.6296E-14	39	5.34989E-09
12	7.92964E-15	26	6.9709E-13	40	1.63177E-09
13	-1.76335E-15	27	4.35009E-14		

From the Table 3, we note that for $|a| \geq 1$, all the partial sums (S_{11}, S_{21}, S_{31}, and S_{41}) are more or less the same, which means adding more terms to the sum has insignificant effect on the value of the sum, implying the series is convergent. However, for $|a| < 1$, the partial sums progressively become larger ($S_{41} > S_{31} > S_{21} > S_{11}$); so, the sum of infinite number of terms tends to infinity, which means the series is divergent.

TABLE 3. Partial sums of terms in K_0 for various values of $|a|$.

$\lvert a \rvert$	S_{11}	S_{21}	S_{31}	S_{41}
1.5	0.995013473	0.995013473	0.995013473	0.995013473
1.2	0.988022294	0.988022294	0.988022294	0.988022294
1	0.97586657	0.97586657	0.97586657	0.975866578
0.9	0.964234296	0.964234296	0.964236432	1.082230751
0.85	0.955950756	0.955950764	0.957864374	928.7584625

4. Numerical example

Consider the quintic equation,

$$x^5 + x = 9.09375,$$

for which one real solution has to be determined using the method proposed here. For this purpose, we obtain the infinite series expression from (2.5), and a polynomial equation from (2.6). The ultraradicals, K_0, K_1, K_2, K_3, and K_4 are determined from (2.9) as:

$$K_0 = 0.999996266\ldots, \quad K_1 = -6.047824804\ldots, \quad K_2 = -0.079999488\ldots,$$
$$K_3 = -0.047999629\ldots, \quad K_4 = -0.033599719\ldots,$$

and the coefficients of quartic equation (3.1) are obtained as follows:

$$\frac{K_0}{K_4} = -29.7620421\ldots, \quad \frac{K_1}{K_4} = 179.9962886\ldots,$$
$$\frac{K_2}{K_4} = 2.380957018\ldots, \quad \frac{K_3}{K_4} = 1.428572306\ldots.$$

Using these coefficients, the quartic equation (3.1) is solved using the methods available in literature (see [12, 13]); and the real solution—that is positive and less than unity—is picked as: $(x_1/a) = 0.1649484536\ldots$, and the real solution of the given quintic equation, $x^5 + x = 9.09375$, is then obtained as: $x_1 = 1.50000000000009\ldots$, while the exact solution is 1.5; the difference between the exact value and the determined value is due to the limited number of terms used in the infinite series in (2.5). In this case we have used 15 terms.

Let us use Newton method for finding a real root of the given polynomial, $f(x) = x^5 + x - 9.09375$, numerically. The Newton method provides an expression as given below for the iterative procedure to obtain a better approximation (x_{n+1}) to the root than the previous approximation (x_n).

$$x_{n+1} = x_n - [f(x_n)/f'(x_n)]$$

The initial guess for the root is denoted as x_0, which has to be a proper guess. In the above example, since we know that the value of the real root is greater than one, let $x_0 = 1$. Using the iterative procedure mentioned above, an approximate real root is obtained in seven iterations, which has an accuracy better than that obtained from the proposed method.

Concluding comments

This paper has presented a new method to obtain a real solution of a single-parameter Bring quintic equation, $x^5 + x = a$, where a is real. The method transforms the quintic equation to an infinite but convergent series, which is further transformed into a quartic equation in a novel fashion, whose coefficients are some kind of infinite series functions (ultraradicals) in a^{-4}. Thus the task of solving the quintic equation reduces now to solving a quartic equation. The proposed method can be used for values of $|a| > 1$, as the series functions in the ultraradicals converge for $|a| > 1$.

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SPACES WITH $\mathscr{M}$-STRUCTURES

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Abstract. In this paper, we introduce the notion of $\mathscr{M}$-structures and study some properties of spaces endowed with $\mathscr{M}$-structures. We see that there is a $\mathscr{M}$-structure in every infinite set.

1. Introduction

Let X be a non-empty set. By a proper subset A of X we mean that A is a non-empty subset of X such that $A \neq X$ and in this case we write $A \subsetneq X$.

It is well known to us that $\{\emptyset\} \cup \{(a,b) : a,b \in \mathbb{R}, a \neq b\}$ forms a basis for the real number space $\mathbb{R}$. The collection $\mathscr{A} = \{(a,b) : a,b \in \mathbb{R}, a < b\}$ of proper subsets of $\mathbb{R}$ admits a special character: for any $A \in \mathscr{A}$ there exist $B, C \in \mathscr{A}$ such that $B \subsetneq A \subsetneq C$. Furthermore, if X is a T_1 connected topological space, then $\{\emptyset\} \cup \mathscr{T}_{mo}$ forms a basis (Theorem 2.4) satisfying the condition that for any $B \in \mathscr{M}$ there exist $A, C \in \mathscr{M}$ such that $A \subsetneq B \subsetneq C$, where $\mathscr{T}_{mo}$ is the collection of all mean open sets in X. Considering these facts, we develop a new kind of structure (resp., space) in nonempty sets namely $\mathscr{M}$-structures (resp., $\mathscr{M}$-space) (Definition 3.1). In recent years Smarandache multispace theory becomes a centre of attraction. Mao [3, 4, 5, 6] studied the Smarandache multispace theory significantly. Under the light of the Smarandache multispace theory, one can say that the study of $\mathscr{M}$-spaces is a particular case sudy of Smarandache multispaces.

2. Preliminaries

Firstly, we recall the following definitions and results:

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Definition 2.1 (Nakaoka and Oda [9, 10, 11])**.** A nonempty open set U of a topological space X is said to be a minimal open set if and only if any open set which is contained in U is $\emptyset$ or U.

Definition 2.2 (Mukharjee and Bagchi [7])**.** An open set M of a topological space X is said to be a mean open if there exist two distinct proper open sets U, V such that $U \subsetneq M \subsetneq V$.

Definition 2.3 (Benchalli et al. [2])**.** A topological space X is said to be a T_{min} space if every proper open set of X is minimal open.

Theorem 2.4 (Nakaoka and Oda [9])**.** *If U is a minimal open set and W is an open set of a topological space X, then either $U \cap W = \emptyset$ or $U \subset W$. If W is a minimal open set distinct from U, then $U \cap W = \emptyset$.*

Theorem 2.5 (Bagchi and Mukherjee [1])**.** *Let $(X, \mathscr{T})$ be a T_1 connected topological space and $\mathscr{T}_{mo}$ denotes the family of all mean open sets in X. Then $\mathscr{B} = \{\emptyset\} \cup \mathscr{T}_{mo}$ forms a basis of the topology $\mathscr{T}$ on X.*

3. $\mathscr{M}$-SPACES

Definition 3.1. Let X be a non-empty set. A collection $\mathscr{A}$ of proper subsets of X is said to be an $\mathscr{M}$-structure on X if for any $A \in \mathscr{A}$ there exist $B, C \in \mathscr{A}$ such that $B \subsetneq A \subsetneq C$. The ordered pair $(X, \mathscr{A})$ is said to be an $\mathscr{M}$-space.

Example 3.2. Let all the proper open sets of a topological space $(X, \mathscr{T}$ be mean open. We write $\mathscr{M} = \mathscr{T} - \{\emptyset, X\}$. Then $(X, \mathscr{M})$ is an $\mathscr{M}$-space.

Remark 3.3. $\mathscr{A} = \{(a, b) : a, b \in \mathbb{R}, a, b\}\}$ and $\mathscr{B} = \{[a, b] : a, b \in \mathbb{R}, a < b\}$ are $\mathscr{M}$-structures on $\mathbb{R}$. Here $(1, 2), (2, 3) \in \mathscr{R}$ but $(1, 2) \cup (2, 3) \notin \mathscr{R}$. On the other hand $[1, 2], [2, 3] \in \mathscr{B}$ but $\{2\} = [1, 2] \cap [2, 3] \notin \mathscr{B}$. Therefore $\mathscr{M}$-structures may not closed under unions as well as intersections.

Theorem 3.4. *Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space. Then each member of the $\mathscr{M}$-structure $\mathscr{A}$ is infinite.*

Proof. Let $A \in \mathscr{A}$. If possible, let A be finite and $A = \{a_1, a_2, ..., a_n\}$, for some natural number $n \geq 1$. Then there is a $A_1 \in \mathscr{A}$ such that $A_1 \subsetneq A$. So $A_1 \subseteq A - \{a_{j_1}\}$, for some $j_1 \in \{1, 2, ..., n\}$. Again there is a $A_2 \in \mathscr{A}$ such that $A_2 \subsetneq A_1$. Thus $A_2 \subseteq A - \{a_{j_1}, a_{j_2}\}$, for some $j_2 \in \{1, 2, ..., n\}$ with $j_1 \neq j_2$. Continuing the process we can have $A_{n-1} \in \mathscr{A}$ such that $A_{n-1} \subseteq$

$A - \{a_{j_1}, a_{j_2}, ..., a_{j_{n-1}}\}$, where $j_k \in \{1, 2, ..., n\}$ with $j_1 \neq j_2 \neq ... \neq j_{n-1}$ and $k = 1, 2, ..., n-1$. Thus either A_{n-1} is a singleton set or $A_{n-1} = \emptyset$. Thus there is no $B \in \mathscr{A}$ such that $B \subsetneq A_{n-1}$, which contradicts $A_{n-1} \in \mathscr{A}$. So A is infinite. Since $A \in \mathscr{A}$ is arbitrary, each member of the $\mathscr{M}$-structure $\mathscr{A}$ is infinite. □

Theorem 3.5. *Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space. Then $\mathscr{A}$ is an infinite collection of proper subsets of X.*

Proof. If possible, let $\mathscr{A} = \{A_1, A_2, ..., A_n\}$ for some natural number $n \geq 1$. Since $A_1 \in \mathscr{A}$, there is a $B \in \mathscr{A} - \{A_1\}$ such that $A_1 \subsetneq B$. Now after some finite steps we can have a $C \in \mathscr{A} - \{A_1, B\}$ such that there is no $D \in \mathscr{A}$ such that $C \subsetneq D$. Thus $\mathscr{A}$ is an infinite collection of proper subsets X. □

Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space. Then X is infinite.

Proof. The proof follows from the fact that X has infinite subsets. □

Theorem 3.6. *Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space. There exist $\mathscr{M}$-structures $\mathscr{B}$ and $\mathscr{C}$ such that $\mathscr{A} \neq \mathscr{B} \neq \mathscr{C}$. In other words, an $\mathscr{M}$-space contains at least three $\mathscr{M}$-structures.*

Proof. Let $\mathscr{B} = \{X - A : A \in \mathscr{A}\}$ and $B \in \mathscr{B}$. Then $B = X - A$ for some $A \in \mathscr{A}$. There exists $A_1, A_2 \in \mathscr{A}$ such that $A_1 \subsetneq A \subsetneq A_2$. So $X - A_2 \subsetneq X - A \subsetneq X - A_1$, i.e, $X - A_2 \subsetneq B \subsetneq X - A_1$. Furthermore $X - A_1, X - A_2 \in \mathscr{B}$. Thus $\mathscr{M}$ is an $\mathscr{M}$-structure on X different from $\mathscr{A}$. One can easily prove that $\mathscr{C} = \{A \subsetneq X : A \in \mathscr{A} or A \in \mathscr{B}\}$ is an $\mathscr{M}$-structure on X which is different from $\mathscr{A}$ as well as $\mathscr{B}$. □

Remark 3.7. Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space. An $\mathscr{M}$-structure $\mathscr{B}$ on X is said to be conjugate to $\mathscr{A}$ iff $\mathscr{B} = \{X - A : A \in \mathscr{A}\}$. In this case, we write $\mathscr{B} = \mathscr{A}^c$. Furthermore, the $\mathscr{M}$-structures $\mathscr{A}$ and $\mathscr{B} = \mathscr{A}^c$ are said to be conjugate to each other.

Theorem 3.8. *There exists $\mathscr{M}$-spaces.*

Proof. Let X be an infinite set. We consider the collection $\mathscr{A} = \{A \subsetneq X : A$ and $X - A$ both are infinite$\}$. Now let $A \in \mathscr{A}$. Then both A and $X - A$ are infinite proper subsets of X. There are points $x \in A$ and $y \in X - A$ such that $A - \{x\} \subsetneq A \subsetneq A \cup \{y\}$. By the definition of $\mathscr{A}$, $A - \{x\}$ and $A \cup \{y\}$ are members of $\mathscr{A}$. So $\mathscr{A}$ is an $\mathscr{M}$-structure on X, i.e., $(X, \mathscr{A})$ is a $\mathscr{M}$-space. □

The $\mathscr{M}$-structure $\mathscr{A}$ defined on an infinite set X discussed on the previous theorem is said to be the trivial $\mathscr{M}$-structure and the $\mathscr{M}$-space $(X, \mathscr{A})$ is said to be the trivial $\mathscr{M}$-space.

Definition 3.9. Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space and $M \subseteq X$. M is said to be an $\mathscr{M}$-set Of X if there exist $A, B \in \mathscr{A}$ such that $A \subsetneq M \subsetneq B$.

If M is an $\mathscr{M}$-set then $M \neq \emptyset, X$. Clearly if $A \in \mathscr{A}$ then A is an $\mathscr{M}$-set.

We denote the collection of all $\mathscr{M}$-sets of X by $\mathscr{M}^*$. One can easily verify that $\mathscr{M}^*$ is an $\mathscr{M}$ structure on X. If $\bigcup_{A\in\mathscr{A}} A = X$, then $\mathscr{M}^*$ is an s-refinement ([8]) of $\mathscr{A}$.

Example 3.10. Let us consider the $\mathscr{M}$-space $(\mathbb{R}, \mathscr{A})$, where $\mathscr{A} = \{(a, b) : a < b \text{ and } a, b \in \mathbb{R}\}$. If M is a countable subset of $\mathbb{R}$, then M is not a $\mathscr{M}$-set. Again for any $a, b \in \mathbb{R}$ with $a < b$, $(a, b]$ and $[a, b)$ are $\mathscr{M}$-sets.

Now let $(X, \mathscr{A})$ be a $\mathscr{M}-space$ and M be a $\mathscr{M}$-set. Then $\{P \in \mathscr{A} : P \subsetneq A\}$ and $\{P \in \mathscr{A} : A \subsetneq P\}$ are nonempty collection of nonempty subsets of X. We write $M_L = \bigcup\{P \in \mathscr{A} : P \subsetneq A\}$ and $M_R = \bigcap\{P \in \mathscr{A} : A \subsetneq P\}$. Clearly $M_L \subseteq M \subseteq M_R$. We call M_L and M_R are the left variation and right variation of the $\mathscr{M}$-set M respectively and $M_R - M_L$ is said to be the variation of the $\mathscr{M}$-set M. We denote the variation of an $\mathscr{M}$-set M by $v(M)$.

Let ρ be the relation on $\mathscr{M}^*$ defined by: "$M\rho N$ if and only if $v(A) = v(B)$, for any $M, N \in \mathscr{M}^*$". We can prove that ρ is an equivalence relation on $\mathscr{M}^*$.

Example 3.11. Let $X = \mathbb{R}^2$ and $\mathscr{A} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq n^2, n \in \mathbb{N}\} \bigcup \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1/n^2, n \in \mathbb{N}-\{1\}\}$. Then $\mathscr{A}$ is an $\mathscr{M}$-structure on $\mathscr{R}^2$. Let $M = \{(x, y) \in \mathbb{R} : x^2 + y^2 = 1\}$. Then $M_L = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1/4\}$ and $M_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

Theorem 3.12. *Let $(X, \mathscr{A})$ be an $\mathscr{M}$-space and $M, N \in \mathscr{A}$ be such that $M \subseteq N$. Then*

(i) $M_L \subseteq N_L$*; and*
(ii) $M_R \subseteq N_R$.

Proof. (i) Let $x \in M_L$. Then $x \in P$ for some $P \in \mathscr{A}$ with $P \subsetneq M$. Since $M \subseteq N$, it follows that $P \subsetneq B$ and so $x \in B_L$. Thus $A_L \subseteq B_L$.

(ii) If $P \in \{P \in \mathscr{A} : N \subsetneq P\}$ then $P \in \{P \in \mathscr{A} : M \subsetneq P\}$, since $M \subseteq N$. Therefore $\{P \in \mathscr{A} : N \subsetneq P\}$ is a subcollection of $\{P \in \mathscr{A} : M \subsetneq P\}$ and thus $\bigcap\{P \in \mathscr{A} : M \subsetneq P\} \subseteq \bigcap\{P \in \mathscr{A} : N \subsetneq P\}$, i.e., $M_R \subseteq N_R$.

□

Theorem 3.13. *Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space and M be a $\mathscr{M}$. Then*

(i) $M \nsubseteq v(M)$;

(ii) $v(M) \subsetneq M$ iff $M = M_R$.

Proof. (i) $M \subseteq v(M) \Rightarrow M \subseteq M_R - M_L \Rightarrow M \subseteq X - M_L \Rightarrow M_L \subseteq X - M_L$, which is a contradiction.

(ii) $M = M_R \Rightarrow v(M) = M - M_L \Rightarrow V(M) \subseteq A$. Using (i) we have $v(M) \subsetneq M$.

Now $v(M) \subsetneq M \Rightarrow M_R \cap (X - M_L) \subsetneq M \Rightarrow M_L \cup [M_R \cap (X - M_L)] \subseteq M_L \cup M = M \Rightarrow (M_L \cup M_R) \cap [M_L \cup (X - M_L)] \subsetneq M \Rightarrow M_R \cap X \subsetneq M \Rightarrow M_R \subsetneq M \subseteq M_R \Rightarrow M = M_R$.

□

Theorem 3.14. *Let $(X, \mathscr{A})$ be the trivial $\mathscr{M}$-space. Then:*

(i) $\mathscr{A} = \mathscr{M}^*$; and

(ii) $v(M) = \emptyset$, for each $M \in \mathscr{A}$.

Proof. (i) It is sufficient to prove that if $M \in \mathscr{M}^*$, then $M \in \mathscr{A}$ for each $M \in \mathscr{M}^*$. Let $M \in \mathscr{M}^*$. Then there exist $A, B \in \mathscr{A}$ such that $A \subsetneq M \subsetneq B$. As $A \subsetneq M$ and A is infinite, so M is also infinite. Now $M \subsetneq B \Rightarrow X - B \subsetneq X - M$. Since $X - B$ is infinite, it follows that $X - M$ is infinite. Thus $M \in \mathscr{A}$. Therefore $\mathscr{A} = \mathscr{M}^*$.

(ii) Let $M \in \mathscr{A} = \mathscr{M}^*$. So M and $X - M$ are infinite proper subsets of X. We can choose distinct points $m, n \in M$ such that $M - \{m\} \subseteq M_L$ and $M - \{n\} \subseteq M_L$. Now $(M - \{m\}) \cup (M - \{n\}) = M$ and so $M = M_L$. On the other hand we can choose two distinct points $p, q \in X - M$ such that $M_R \subseteq M \cup \{p\}$ as well as $M_R \subseteq M \cup \{q\}$. So $M = (M \cup \{p\}) \cap (M \cup \{q\})$ and thus $M = M_R$. Hence $v(M) = M_R - M_L = M - M = \emptyset$. Since $M \in \mathscr{A}$ is arbitrary, $v(M) = \emptyset$, for each $M \in \mathscr{A}$.

□

Definition 3.15. Let $M \subsetneq X$. M is said to be a common $\mathscr{M}$-set of X if M is an $\mathscr{M}$-set of X with respect to $\mathscr{A}$ as well as an $\mathscr{M}$-set of X with respect to $\mathscr{A}^c$.

Theorem 3.16. *Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space. Then followings are equivalent:*

(i) M is a common $\mathscr{M}$-set of X.
(ii) M and $X - M$ are $\mathscr{M}$-sets with respect to $\mathscr{A}$.
(iii) M and $X - M$ are $\mathscr{M}$-sets with respect to $\mathscr{A}^c$.

Proof. $(i) \Rightarrow (ii)$:
There exist $A, B \in \mathscr{A}$ and $C, D \in \mathscr{A}^c$ such that $A \subsetneq M \subsetneq B$ and $C \subsetneq M \subsetneq D$. By the definition of $\mathscr{A}^c$, $C = X - A_1$ and $D = B_1$, for some $A_1, B_1 \in \mathscr{A}$. Then $B_1 \subsetneq X - M \subsetneq A_1$. Thus M and $X - M$ are $\mathscr{M}$-sets with respect to $\mathscr{A}$.

$(ii) \Rightarrow (iii)$:
There exist $A, B, C, D \in \mathscr{A}$ such that $A \subsetneq M \subsetneq B$ and $C \subsetneq X - M \subsetneq D$. Now $X - A, X - B \in \mathscr{A}^c$ and $X - B \subsetneq X - M \subsetneq X - A$. Also $X - C, X - D \in \mathscr{A}^c$ such that $X - D \subsetneq M \subsetneq X - C$. Thus M and $X - M$ are $\mathscr{M}$-sets with respect to $\mathscr{A}^c$.

$(iii) \Rightarrow (i)$:
There exist $A, B, C, D \in \mathscr{A}$ such that $X - A \subsetneq M \subsetneq X - B$ and $X - C \subsetneq X - M \subsetneq X - D$. Then $D \subsetneq M \subsetneq C$ and so M is a common $\mathscr{M}$-set of X. □

Theorem 3.17. *Let $(X, \mathscr{A})$ be a $\mathscr{M}$-space and M be a common $\mathscr{M}$-set of X. Then followings are true:*

(i) There exist $A, B \in \mathscr{A}$ such that $A \cap B = \emptyset$ and $A \cup B = X$.
(ii) There exist $C, D \in \mathscr{A}^c$ such that $C \cap D = \emptyset$ and $C \cup D = X$.

Proof. $(i) \Rightarrow (ii)$:
By the previous Theorem, we choose $A = M$ and $B = X - M$.

$(ii) \Rightarrow (iii)$:
By the previous Theorem, we choose $C = M$ and $D = X - M$. □

4. Remark on T_{min} spaces

Let X be a T_{min} space. Then all the proper open sets of X are minimal open sets. By Theorem 2.4, all the proper open sets of X mutually disjoint. We claim that X can have atmost two proper open sets. In fact, if X has more than two proper open sets then union of any two proper open sets must be a proper open set containing two proper open sets (since all the proper open sets are mutually disjoint). Consequently, X has a proper open set which is not a minimal open set, but this contradicts the fact that X

is a T_{min} space. On the other hand, if a topological space X has only one proper open set then X must be a T_{min} space. Further more, if a topological space X has only two disjonit proper open sets then the proper open sets must be minimal, i.e., X must be a T_{min} space.

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ON FACTORS OF $\Phi_P(M)$ IN $\mathbb{F}_2[X]$ AND SELF-RECIPROCAL POLYNOMIALS

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Abstract. We prove in a few new cases (under conditions related to self-reciprocal polynomials) the non-existence of Mersenne primes M such that the composite polynomial $\Phi_p(M)$ is a product of other Mersenne primes, where $\Phi_p(x) \in \mathbb{F}_2[x]$ is the p-th cyclotomic polynomial.

1. Introduction

Given an irreducible polynomial $f(x)$ over a field $\mathbb{K}$, given a polynomial (or a rational fraction) $g(x) = C(x)/D(x)$ over the same field. It is natural to consider the behaviour of the composite polynomial $F(x) = D(x)^{\deg(f(x))} \cdot f(g(x))$ that come from properties of $f(x)$ and $g(x)$ (see [1, 2, 3, 4, 5, 6, 8, 9, 18, 19, 20, 21, 22, 23, 27, 28, 29, 31, 32, 34]).

Generally, $g(x)$ has been chosen as either a power of x, as a *linearized* polynomial, e.g., $g(x) = x^{p^r} - x$, with p the characteristic of $\mathbb{F}_q$, or as a quotient A/B of two other polynomials, sometimes both A and B of degree ≤ 1, such that $g(x)$ is one to one. For instance, Panario et al. [27] took $A := 1$, and $B := cx + 1$, with nonzero $c \in \mathbb{F}_q$ in order to obtain conditions such that $f(x)$ irreducible implies $F(x)$ irreducible. This property also appears in, e.g., [9, 29, 30]. Most classical results about polynomials over finite fields appear in Lidl and Niederreiter [26], or Swan [33].

In the present paper we take for $g(x)$ a binary polynomial, motivated by the fact that we may consider the ring $\mathbb{F}_2[x]$ as one of the *closest* analogue of the ring of integers $\mathbb{Z}$. We might think that $2^{a+b} \in \mathbb{Z}$ corresponds

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(up to switching x and $x+1$) to $x^a(x+1)^b \in \mathbb{F}_2[x]$. We take for $f(x)$ the cyclotomic polynomial $\Phi_p(x) = (x^p+1)/(x+1) \in \mathbb{F}_2[x]$, with p a prime number. One might say that we chose $f(x), g(x)$ as *arithmetic* polynomials.

A few words on notations used throughout the paper: if $A \in \mathbb{F}_2[x]$ is irreducible then we say that A is *prime*, a *Mersenne* polynomial $M \in \mathbb{F}_2[x]$ (an analogue of a Mersenne number by the correspondance above) is a polynomial such that $M+1$ is a product of powers of x and powers of $x+1$. We say that $M+1$ *splits.* When a Mersenne polynomial M is irreducible, we say that M is a *Mersenne prime.* A binary polynomial A is *complete* [7], if all coefficients of A are equal to 1. A binary polynomial B is *odd* if $B(0) = B(1) = 1$, otherwise B is *even.* More standard notation follows. The *reciprocal* $A^*(x)$ of a polynomial $A(x)$ of degree n, with $A(0) = 1$, is defined by $A^*(x) = x^n A(1/x)$. A polynomial is called *self-reciprocal* if it coincides with its reciprocal. Let $\omega(P)$ denote the number of pairwise distinct prime factors of $P \in \mathbb{F}_q[x]$, $o_p(b)$ the multiplicative order of a nonzero element of the finite field $\mathbb{F}_p$, $o(\alpha)$ the multiplicative order of an element α in some appropriate algebraic extension of $\mathbb{F}_2$; $ord(H)$ the minimal positive integer m such that the binary polynomial H divides the binomial $x^m - 1$. We let $A'(x)$ denote the formal derivative of $A(x)$ relative to x. Finally, we let $\overline{\mathbb{F}_2}$ denote a fixed algebraic closure of $\mathbb{F}_2$.

One easily check that $\Phi_p(x)$ is square-free, but very little information [26, Theorem 2.47] is available about its *prime* (i.e., irreducible) factors. Let M be a Mersenne prime, we know [15, Lemma 2.6] that $\Phi_p(M)$ is square-free. The special equation that we consider, in which we take $f(x) = \Phi_p(x)$ and $g(x) = M$, with M a Mersenne prime, is the following

$$\Phi_p(M) = M_1 \cdots M_s. \tag{1.1}$$

with, $M_1, \ldots, Ms$ Mersenne primes, and, say, $\deg(M_1) \leq \cdots \leq \deg(M_s)$.

The study of equation (1.1) might have some interest since very few is known about the prime factors of composite polynomials, [9, 20, 27, 29, 32, 34], in particular for polynomials of the form $\Phi_p(A(x))$.

It appears that the solutions of the equation (1.1) are linked (see [7, 10, 12, 13, 14, 15, 16, 17]) to the existence of *binary perfect* polynomials. Briefly, a binary perfect polynomial is a fixed point of a function analogue to the classical sum of divisors function over the integers. Over $\mathbb{F}_2[x]$, $\sigma(A) = \sum_{D|A} D \in \mathbb{F}_2[x]$ is the sum of all divisors of A in $\mathbb{F}_2[x]$, including the trivial divisors 1 and A. One has $\sigma(XY) = \sigma(X)\sigma(Y)$ when $\gcd(X, Y) = 1$. Observe that $\sigma(Q^p) = \Phi_p(Q)$ for a prime number $p \in \mathbb{Z}$ and a prime polynomial $Q \in \mathbb{F}_2[x]$. This σ function is more natural, but also more complicated, that the usual sum of divisors function $\sigma_1 : \mathbb{F}_2[x] \mapsto \mathbb{R}$, defined by $\sigma_1(A) = \sum_{D|A} 2^{\deg(A)}$, for any $A \in \mathbb{F}_2[x]$. Besides the perfect polynomials 0 and 1, and the polynomials $T(n) = (x(x+1))^{2^n-1}$, $n \geq 1$, called (*trivial*) perfect, there exist 11 known non-trivial binary perfect polynomials, called (*sporadic*) perfect. The link is that equation (1.1) implies a characterization of 9 [15, 16, 17] of these 11 sporadic perfect, as the only even binary perfect polynomials, all of whose odd prime divisors, i.e., prime divisors that are coprime with $x(x + 1)$, are Mersenne primes.

The binary perfect polynomials are a polynomial analogue of the multiperfect numbers over $\mathbb{Z}$. One has that for $A \in \mathbb{F}_2[x]$, $\sigma(A)/A \in \mathbb{F}_2[x]$ is equivalent to $A = \sigma(A)$. Canaday, the first Ph. D. student of Leonard Carlitz, start the work on the binary perfect polynomials in 1941 in [7], (a simplified version of his Ph.D. thesis).

The equation (1.1) seems very difficult to resolve in general. However, when $\Phi_p(M)$ has some divisors that are self-reciprocal, something can be said. The contribution of the present paper consists of a simple study of some of these cases (see Theorem 1.1) that generalize the only known example in equation (1.3). We give an upper bound for s when no M_j is self-reciprocal (see Proposition 1.2). We also consider the special case in which M is also the reciprocal polynomial of a Selmer trinomial. More precisely, we take $M := x^c(x + 1) + 1$, the reciprocal polynomial of the Selmer trinomial $T := x^{c+1} + x + 1$. With the same M, we obtain new results in the unsolved (and apparently, non-trivial) case $p = 5$.

Our motivation for focusing in possible self-reciprocal divisors of $S := \Phi_p(M)$, comes from the well known facts that $\Phi_n(x)$ is self-reciprocal, and the binomial $B := x^{ord(S)} - 1$ in $\mathbb{F}_2[x]$ is a multiple of S, and factors [26, Theorem 2.47] as:

$$x^{ord(S)} - 1 = \prod_{n \mid ord(S)} \Phi_n(x). \tag{1.2}$$

Our main result is as follows:

Theorem 1.1. *Let p be an odd prime number such that 2 is a primitive root modulo p, and let s be a positive integer. Let $M := x^a(x+1)^b+1 \in \mathbb{F}_2[x]$ be a Mersenne prime, and $T := M+1$. For $j = 1, \ldots, s$, let $M_j := x^{a_j}(x+1)^{b_j}+1$ be a Mersenne prime. Let $k_j := \deg(M_j)/o_p(2)$. The equation (1.1), i.e.,*

$$1 + M + \cdots + M^{p-1} = M_1 \cdots M_s.$$

is impossible in each of the cases (a), $\cdots$, (e) besides the following known case (up to switching of x and $x+1$): $p = 3$, $s = 2$, $M = x(x+1)^2 + 1 = x^3 + x + 1$, $M_1 = x(x+1) + 1$, and $M_2 = x^3(x+1) + 1$; namely,

$$1 + M + M^2 = (x(x+1) + 1)(x^3(x+1) + 1). \tag{1.3}$$

Where:

- (a) *One has $M_1 = x^2 + x + 1$, and for a root $\alpha \in \overline{\mathbb{F}_2}$ of M_1 one has $M(\alpha) = \alpha$.*
- (b) *One has $M_2 = x^4 + x^3 + 1$, and $M = x^3 + x + 1$.*
- (c) *One has that M_s has even degree $2m$, the order $e := ord(M_s)$ divides $2^m + 1$, and e does not divide $2^k + 1$ for any $0 \le k < m$.*
- (d) *The polynomial $S := \Phi_p(M)$ has a self-reciprocal divisor D with $\deg(D) \ge \deg(S)/2$.*
- (e) *Assume that for some $n \mid ord(S)$ with $S := \Phi_p(M)$, there exists an $w > 0$ with $2^w \equiv -1 \pmod{n}$.*
- (f) *Assume that $f(x) \in \mathbb{F}_2[x]$ satisfies the following:*
  - *$M_s \mid f(x)$, and*
  - *$\deg(f(x)) = 4m$, and*
  - *$ord(f(x)) = 2^{2m} + 1$, and that for some $n \mid ord(M_s)$ one has*
  - *$f(x) \mid \Phi_n(x)$.*

The following proposition follows from [11, Theorem 2.5].

Proposition 1.2. *If no $M_1, \ldots, M_s$ is self-reciprocal in equation (1.1), then*

$$s \le \sum_{d \mid n,\ 2^w \not\equiv -1 \pmod{d},\ for\ all\ w>0} \varphi(d)/o_d(2).$$

where $n = ord(\Phi_p(M))$, and φ is the Euler function.

Proof. We choose $n_1 := 1$, $n_2 := n$, $\lambda := 1$, $r := 1$, and $q = 2$ in [11, Theorem 2.5]. This gives, with the notations of the cited theorem, $s \leq N_1 - N_2$. That is, the result. □

Remark 1.3. One checks that the conditions in Theorem 1.1 (besides (f)) generalize the known case in equation (1.3). Thus, we may think that these conditions are specific to the known case, in equation (1.3), and cannot be extended to more general cases.

As sketched above, we have also a new result in a special case, in which p is an arbitrary odd prime, but the Mersenne prime M is the reciprocal of a classical Selmer trinomial $x^n + x + 1$. We focus on the case in which (furthermore) $p = 5$. One reason for this is that the case $p = 5$ is exceptional. Indeed, for any other Fermat prime $p > 5$, equation (1.1) is impossible, essentially by a result of Swan [33] about reducible trinomials (see [16, Corollary 3.3], and [17, Corollary 3.26]). But for the Fermat prime 5 (and any Mersenne prime M) we do not known whether or not equation (1.1) holds.

Our result for the case $p = 5$ is the following.

Theorem 1.4. *Let $M := x^{c+1} + x^c + 1 \in \mathbb{F}_2[x]$ be prime. Let $p = 5$, and $r = \omega(\Phi_p(M))$. Then:*

(a) *One has that $c > 1$ is odd if and only if r is even.*
(b) *For c from 2 to 21844, equation (1.1) does not hold.*
(c) *If equation (1.1) holds with $p = 5$, then $r > 3$.*
(d) *If equation (1.1) holds with $p = 5$, $c > 1$ is odd, $\Phi_p(x) \mid \Phi_p(M)$, and $\Phi_p(x) \neq M_1$, then $r \geq 6$.*

Remark 1.5. We computed all c's up to 100000, for which M is prime (see Lemma 2.13). However, we had not enough available RAM to be able to test equation (1.1) for $c \geq 21845$. Moreover, (see (d)), we left open the case in which $\Phi_p(x) = M_1$. These case is more technically complicated, since we have now 4 unknowns to deal with, namely, M, M_2, M_3 and M_4.

2. Tools

Although well known, the following lemma is very useful, since we do not know the exact form of the prime factors of $\Phi_p(x)$ in $\mathbb{F}_2[x]$ (we know, however, how many they are, and which degrees they have [26, Theorem 2.47]).

Lemma 2.1. *Let p be an odd prime number. The cyclotomic polynomial $\Phi_p(x) \in \mathbb{F}_2[x]$ is irreducible if and only if 2 is a primitive element of the finite field $\mathbb{F}_p$.*

Observe that it is believed, but not yet proved, that there are an infinity of such prime numbers.

The following lemma is a particular case of [28, Satz 5] and it is easy to check.

Lemma 2.2. *Let $f(x) \in \mathbb{F}_2[x]$ be a prime polynomial of degree k, and let $g(x) \in \mathbb{F}_2[x]$. Let $F(x) = f(g(x))$. Then, the degree of every prime divisor of $F(x)$ is divisible by k.*

The following lemma follows from Lemma 2.2.

Lemma 2.3. *Let p be a prime number such that $o_p(2) = p - 1$. Let $g(x) \in \mathbb{F}_2[x]$ be such that*

$$\Phi_p(g(x)) = m_1(x) \cdots m_s(x) \tag{2.1}$$

for s pairwise distinct prime polynomials in $\mathbb{F}_2[x]$. Then for $j = 1, \ldots, s$, $p - 1$ divides $\deg(m_j(x))$, and

$$\deg(g(x)) = k_1 + \cdots + k_s, \tag{2.2}$$

where $k_j := \deg(m_j(x))/(p - 1)$.

The following result follows from [11, Theorem 2.3], [23, 25]:

Lemma 2.4. *Let $f(x) \in \mathbb{F}_2[x]$ be an irreducible polynomial such that for some odd n, $f(x) \mid \Phi_n(x)$. The following holds: $f(x)$ is self-reciprocal if and only if there exists an $w > 0$ such that*

$$2^w + 1 \equiv 0 \pmod{n}. \tag{2.3}$$

Roughly, this comes from the fact that if $f(x)$ is self-reciprocal then, by Galois theory, $f(x)$ has a zero $\beta \in \overline{\mathbb{F}_2}$ such that, $1/\beta = \beta^{2^k}$ for some k.

The following lemma of Mironchikov [24] (see also [26, p. 133]) is also useful.

Lemma 2.5. *Let $f(x) \in \mathbb{F}_2[x]$ be with $\deg(f(x) = 2m$, m even, and such that $ord(f(x)) = 2^m + 1$. Then $f(x)$ is irreducible in $\mathbb{F}_2[x]$.*

The following three lemmas are Canaday [7, Lemma 8, Theorem 8, Corollary to Theorem 8].

Lemma 2.6. *If $P \in \mathbb{F}_2[x]$ is a self-reciprocal Mersenne prime, then $P \in \{x^2+x+1, x^4+x^3+x^2+x+1\}$.*

Lemma 2.7. *Let $A \in \mathbb{F}_2[x]$ be a complete polynomial such that every irreducible factor of A is a Mersenne prime. Then $A \in \{x^2+x+1, x^4+x^3+x^2+x+1, (x^3+x+1)(x^3+x^2+1)\}$.*

Lemma 2.8. *Let $A \in \mathbb{F}_2[x]$ be a self-reciprocal polynomial such that every irreducible factor of A is a Mersenne prime. Then*

$$rad(A) \mid B \cdot C, \tag{2.4}$$

where

$$B := (x^2+x+1)(x^3+x+1)(x^3+x^2+1)(x^4+x^3+x^2+x+1), \tag{2.5}$$

and

$$C := (x^9+x^8+1)(x^9+x+1). \tag{2.6}$$

Part (a) of the following lemma follows from [16, Theorem 1.4], while part (b) follows from [17, Theorem 1.4].

Lemma 2.9. *With the notations of Theorem 1.1, assume that equation* (1.1) *holds. Then:*

(a) *One has $s > 2$.*
(b) *One has $\deg(M) > 4$.*

It is also useful to know necessary conditions on $c \geq 1$, provided that $x^{c+1}+x^c+1$ is prime.

The following two lemmas are [3, Theorem 5] and [3, Corollary 3].

Lemma 2.10. *Suppose that the s-degree polynomial $f(x) \in \mathbb{F}_2[x]$ is the product of r pairwise distinct irreducible polynomials over $\mathbb{F}_2[x]$ and, let $F(x) \in \mathbb{Z}[x]$ be any monic lift of $f(x)$ to the integers. Then* $\mathrm{discrim}(F) \equiv 1$ *or* $5 \pmod 8$*, and more importantly, $r \equiv s \pmod 2$ if and only if* $\mathrm{discrim}(F) \equiv 1 \pmod 8$.

Lemma 2.11. *Let $v(x) = x^{2n} + x^{2n-t} + 1, u(x) = x^{2m} + \sum_{j=0}^{2m-1} a_j x^j$ be two polynomials with integer coefficients such that both t and $u(1)$ are odd* *numbers. Furthermore suppose that a_{2m-1} is an even number whenever $n = t$. Then*

$$\operatorname{discrim}(u(v(x))) \equiv \operatorname{discrim}(u(x))^{2n} \pmod 8.$$

The following lemma is useful for computations.

Lemma 2.12. *Let c be a positive integer, and $M(x,c) := x^{c+1} + x^c + 1 \in \mathbb{F}_2[x]$. Assume that $M(x,c)$ is prime. Then:*

(a) *If c is odd, then $c \in \{3,5\} \pmod 8$.*
(b) *If c is even, then: either $c = 2$ or $c \in \{0,6\} \pmod 8$.*
(c) *If for some non-negative integer m, $c = 2^m - 1$ (respectively, $c = 2^m + 1$) then $c \in \{1,3\}$ (respectively, $c \in \{2,3,5\}$).*

Proof. Part (a) follows from [16, Corollary 3.3 (i)] with $a = c$ and $b = 1$. Part (b) follows from [16, Corollary 3.3 (ii)] with $a = 1$ and $b = c$. Part (c) follows from [16, Lemma 4.2] by switching x and $x + 1$. $\square$

Using Lemma 2.12 and a straightforward program in GP-Pari we obtained the following lemma.

Lemma 2.13. *For $c = 1, \ldots, 100000$, the polynomial $M := x^{c+1} + x^c + 1 \in \mathbb{F}_2[x]$ is prime for the following values of c.*

$$1, 3, 5, 6, 8, 14, 21, 27, 29, 45, 59, 62, 126, 152, 171, 302, 470, 531, 864, 899, 1365,$$
$$2379, 3309, 4494, 6320, 7446, 10197, 11424, 21845, 24368, 27285, 28712, 32766,$$
$$34352, 46382, 53483, 62480, 83405, 87381.$$

Remark 2.14. Zierler [35] obtained the values of c up to 30000.

3. Proof of Theorem 1.1

In order to prove (a) observe that equation (1.1) implies that $M(\alpha)^p = 1$, so that α is a zero of $x^p + 1 = (x+1)\Phi_p(x)$. Thus, M_1 is an odd divisor of $(x+1)\Phi_p(x)$, i.e., $M_1 \mid \Phi_p(x)$. But 2 is a primitive root of 1 modulo p, thus $\Phi_p(x)$ is prime. It follows that $M_1 = \Phi_p(x)$. By [26, Theorem 2.47], we obtain $2 = \deg(M_1) = o_2(p)$. Therefore $p \mid 2^2 - 1$, i.e., $p = 3$. This is impossible by [17] (non-trivial proof).

To prove (b), proceeding as before, equation (1.1) implies that $M(\alpha)^p = 1$, for $\alpha \in \overline{\mathbb{F}_2}$ a zero of M_2. Thus, $\alpha M(\alpha) = (\alpha+1)^3$. It follows that

$$(\alpha+1)M(\alpha) = (\alpha+1)^3 + \alpha^3 + \alpha + 1 = \alpha^2. \tag{3.1}$$

On the other hand, $M_2(\alpha) = 0$ implies $\alpha + 1 = 1/\alpha^3$, so that equation (3.1) implies

$$M(\alpha) = \alpha^5. \tag{3.2}$$

Therefore, $o(\alpha) \mid 5p$. But $o(\alpha) = 15$, since M_2 is primitive. Thus, we must have $15 = o(\alpha) = 5p$. Therefore, $p = 3$. This is impossible by [17] if we are not in the known case.

We prove now (c). By Yucas and Mullen [25] M_s is self-reciprocal. By Lemma 2.6 we have $M_s \in \{x^2+x+1, x^4+x^3+x^2+x+1\}$. By Lemma 2.9 we have $s > 2$, thus $M_s = \Phi_5(x)$. Alternatively, by Lemma 2.3 one has $4 \mid \deg(M_s)$, thus $M_s = \Phi_5(x)$. Since M_s is a prime divisor of $\Phi_p(M)$ with the highest possible degree, and the only Mersenne primes of degree 4 are $\{\Phi_5(x), x^4+x^3+1\}$, we obtain $s = 2$. This contradicts Lemma 2.9, thereby proving the result.

In order to prove (d), observe that D is square-free since S is square-free. Thus $D = \mathrm{rad}(D)$. Since D is a product of Mersenne primes, Lemma 2.8 implies that D is a divisor of BC (using notation in Lemma 2.8). But $\deg(BC) = 30$, thus $p\deg(M) = \deg(S) \leq 60$. Thus, by computations (using Lemmas 2.2, 2.6, 2.7, and 2.8) examining all possibilities, we obtain the result.

To prove (e) observe that M_s is an irreducible polynomial that divides $T := x^{ord(S)} - 1$, since $S \mid T$. One has $ord(S) = \mathrm{lcm}(ord(M_1), \ldots, ord(M_s))$ by [26, Theorem 3.9], and each $ord(M_j)$ equals the order of some zero $\beta_j \in \overline{\mathbb{F}_2}$ of M_j in the extension fields $\mathbb{F}_2[\beta_j]$. Thus $ord(M_j) \mid 2^{\deg(M_j)} - 1$. In other words, $ord(S)$ is an odd number. Moreover, (see equation (1.2)), $T = \prod_{n \mid ord(S)} \Phi_n(x)$, thus $M_s \mid \Phi_n(x)$ for some n that divides $ord(S)$. It follows that n is also odd. By the property of w, Lemma 2.4 implies that M_s is self-reciprocal. Therefore, Lemma 2.6, with $P = M_s$, implies the result.

In order to prove (f), observe that Lemma 2.5 implies that $f(x)$ is irreducible in $\mathbb{F}_2[x]$. Thus $f(x) = M_s$. Taking $w = 2m$, the result follows from part (e), since $ord(M_s) \mid ord(S)$.

4. Proof of Theorem 1.4

Let $f(x) = \Phi_5(M)$. Let $F(x)$ be the monic lift of $f(x)$ to the integers. Observe that $s = \deg(f(x)) = 4\deg(M)$. Lemma 2.10 implies that r is even if and only if $\operatorname{discrim}(F(x)) \equiv 1 \pmod 8$. Put $g(x) = M$. Thus, we may assume that c is odd. The proof of part (a) is as follows. Since c is odd, $g(x) = x^{2n} + x^{2n-t} + 1$, with $n = (c+1)/2$ and $g(1) = 2m + 1 = 1 \in \mathbb{F}_2$. From Lemma 2.10 and Lemma 2.11 we have

$$\operatorname{discrim}(\Phi_5(M)) \equiv (\operatorname{discrim}(\Phi_5(x)))^{c+1} \pmod 8. \tag{4.1}$$

Thus, it suffices to prove that the discriminant $\delta_5 \in \mathbb{Z}$ of the cyclotomic polynomial $\Phi_5(x)$ (lifted to $\mathbb{Z}[x]$), namely, $\delta_5 := \operatorname{discrim}(x^4+x^3+x^2+x+1)$, satisfies

$$\delta_5^{c+1} \equiv 1 \pmod 8. \tag{4.2}$$

This works since $c+1$ is even, and by a computation one has: $\delta_5 = 5^3$, and $5^3 \equiv 5 \pmod 8$.

Part (b) follows from a straightforward computation in GP-Pari that lasted 3 days. Since we first computed the possibles values of c for which M is prime (see Lemma 2.13), it was only necessary to check the result up to $c = 11424$, since the next possible $c \leq 10^5$, is $c = 21845$.

We prove now part (c). By [16, Corollary 4.6] one has $r > 1$. A more complicated proof [16, Theorem 1.4] gives $r > 2$. Assume that $r = 3$. Since $o_5(2) = 4 = p - 1$, i.e., since 2 is primitive modulo 5, Lemma 2.3 implies that $\deg(M_j)$ is multiple of 4. Remember, with the notation of Theorem 1.1, that $M_j := x_j^a(x+1)_j^b + 1$. Thus, $4 \mid a_j + b_j$, so that both a_j and b_j are odd, since M_j prime implies that $\gcd(a_j, b_j) = 1$. But, by [16, Corollary 4.9] both $u := a_1 + a_2 + a_3$ and $v := b_1 + b_2 + b_3$ are even. This implies that $r = u + v$ is also even. This contradicts our assumption that $r = 3$. Therefore, $r > 3$.

We prove now part (d). The cases when $\Phi_p(x) = M_4$ or M_3 do not happen since $\deg(M_1) \le \deg(M_2) \le \deg(M_3) \le \deg(M_4)$, and $4 \mid \deg(M_j)$ for all j, by Lemma 2.3, because there are only two Mersenne primes of degree 4, namely $\Phi_5(x)$ and x^4+x^3+1. We can assume then that $M_1 = x^4+x^3+1$ and that $M_2 = \Phi_5(x)$. We have

$$\Phi_5(M) = x^{4c+4} + x^{4c} + x^{3c+3} + x^{3c+2} + x^{3c+1} + x^{3c} + 1. \tag{4.3}$$

Putting $\Phi_5(M) := x^{4c+4} + \sum_1^{4c+3} k_j x^j$ we have then

$$k_{4c+3} = 0, k_{4c+2} = 0, k_{4c+1} = 0, k_2 = 0, k_1 = 0, k_0 = 1. \tag{4.4}$$

Put also

$$M_{34} := M_3M_4 = x^d + \sum_0^{d-1} \ell_j x^j. \tag{4.5}$$

where $d := a_3 + b_3 + a_4 + b_4$, and

$$M_{12} := M_1M_2 = x^8 + x^4 + x^2 + x + 1. \tag{4.6}$$

Write $M_3 = x^{a_3+b_3} + \sum_{j=0}^{a_3+b_3-1} e_j x^j$. We have $M_3' = x^{a_3-1}(x+1)^{b_3-1}((a_3+b_3)x + a_3)$, $M_3 = 1 + x^{a_3}(1 + b_3x + b_3(b_3-1)/2 \cdot x^2 + \cdots + b_3(b_3-1)/2 \cdot x^{b_3-2} + b_3x^{b_3-1} + x^{b_3})$. Thus $M_3'(0)$ equals 0 if $a_3 > 1$ and equals 1 if $a_3 = 1$. Observe that b_3 is odd so that $b_3 = 1$ in $\mathbb{F}_2$. Thus, if $a_3 = 1$ then $e_2 = b_3 = 1$, if $a_3 > 1$ then $e_2 = 0$, if $a_3 > 1$ then $e_1 = 0$, and if $a_3 = 1$ then $e_1 = 1$. By symmetry, with $M_4 = x^{a_4+b_4} + \sum_{j=0}^{a_4+b_4-1} f_j x^j$, one gets $f_0 = 1$, $f_1 = 0$ if $a_4 > 1$ and $f_1 = 1$ if $a_4 = 1$, $f_2 = 0$ if $a_4 > 1$ and $f_2 = 1$ if $a_4 = 1$. We are now ready to compute the last three coefficients of M_{34}. First of all, trivially $\ell_0 = e_0f_0 = 1 \cdot 1 = 1$, then $\ell_1 = e_1 + f_1$ have the following values: 0 when ($a_3 > 1$ and $a_4 > 1$) or ($a_3 = 1$ and $a_4 = 1$), and 1 otherwise. Observe that $e_1 = e_2$ and $f_1 = f_2$ so that $d_2 = \ell_1 + e_1f_1$. We have that e_1f_1 equals 1 when $a_3 = 1$ and $a_4 = 1$, and equals 0 otherwise. This allows us to compute d_2. We obtain that d_2 equals 0 if and only if $a_3 > 1$ and $a_4 > 1$.

Since $\Phi_5(M) = M_{12}M_{34}$, by comparing coefficients, we have $1 = k_0 = \ell_0$, $0 = k_1 = \ell_0 + \ell_1$, and $0 = k_2 = \ell_0 + \ell_1 + \ell_2$. Since $\ell_0 + \ell_1$ equals 0, one has that either ($a_3 > 1$ and $a_4 = 1$) or ($a_3 = 1$ and $a_4 > 1$). On the other hand, we see that $\ell_0 + \ell_1 + \ell_2 = 0$ if and only if $a_3 = 1$ and $a_4 = 1$. Thus, it is impossible to have simultaneously $k_1 = 0$ and $k_2 = 0$. This implies the contradiction that $\Phi_5(M) \ne M_{12}M_{34}$, thereby proving the theorem.

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BOUNDS FOR THE EIGENVALUES OF GALLAI GRAPHS OF SOME GRAPHS

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Abstract. The line graph $L(G)$, of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $L(G)$, if they are adjacent in G. The Gallai graph $\Gamma(G)$ of a graph G, has the edges of G as its vertices and two distinct vertices are adjacent in $\Gamma(G)$ if they are adjacent edges in G, but do not lie on a triangle. The anti-Gallai graph $\Delta(G)$ of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $\Delta(G)$, if they lie on a common triangle in G. In this paper we find bounds for the eigenvalues of Gallai graph of some class of graphs by using the adjacency spectrum of line graph and anti-Gallai graph.

1. Introduction

Let G be a simple graph on n vertices $\{1, 2, ..., n\}$ with an adjacency matrix $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1, & if\ the\ vertex\ \ i\ is\ adjacent\ to\ \ the\ vertex\ \ j \\ 0, & otherwise. \end{cases}$$

Clearly, $A(G)$ is a real symmetric matrix with diagonal entries zero. The eigenvalues of $A(G)$ are the roots of $det(xI - A) = 0$, the characteristic equation of A or G. $A(G)$ has n eigenvalues. The set of eigenvalues of G is called the spectrum of G and is denoted by $spec(G)$ [7]. Since $A(G)$ is a real symmetric matrix all of its eigenvalues are real, which are denoted by $\lambda_1, \lambda_2, ..., \lambda_n$ and can be ordered as $\lambda_1 \geq \lambda_2, ..., \geq \lambda_n$. The energy of a graph G is denoted by E_G and is defined as $E_G = \Sigma_{i=1}^{n} |\lambda_i|$ [9].

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The line graph $L(G)$ [19] of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $L(G)$ if they are adjacent in G. The adjacency spectrum of line graph of many classes of graphs has been studied in [6, 7, 16]. The Gallai graph $\Gamma(G)$ [17] of a graph G has the edges of G as its vertices and two distinct edges are adjacent in $\Gamma(G)$ if they are adjacent in G, but do not lie on a common triangle in G. The anti-Gallai graph $\Delta(G)$ [17] of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $\Delta(G)$, if they lie on a common triangle in G. Clearly the Gallai graph and the anti-Gallai graph are spanning subgraphs of line graph [17]. Though the line graphs are extensively studied in literature, there are only a handful of papers in Gallai and anti-Gallai graphs. Some of the interesting papers on the Gallai and anti- Gallai graphs are [1, 2, 11, 10, 12, 15, 17, 18].

The spectrum of Gallai graph and anti-Gallai graph of some class of graphs are studied in papers [13, 14, 15]. If G is regular then $L(G)$ is regular. Also there is a relation connecting the incidence matrix of G and the adjacency matrix of $L(G)$. These properties helped in studying the spectrum of line graphs. Unfortunatly, this nice behaviour is not present for Gallai and anti-Gallai graphs. In this paper we find some bounds for the eigenvalues of some class of Gallai graphs by using the spectrum of line graph and anti-Gallai graph.

If G is regular then $\Gamma(G)$ need not be regular. But in paper [15], it is found that G is a strongly regular graph then $\Gamma(G)$ is an edge-regular graph. The graph G is said to be *strongly regular* [4] with parameters (n, k, a, c) if the following conditions hold:
(1) each vertex has k neighbours;
(2) any two adjacent vertices of G have 'a' common neighbours;
(3) any two non-adjacent vertices of G have 'c' common neighbours.
An *edge-regular graph* [5] with parameters (n, k, a) is a graph on n vertices which is regular of degree 'k' and such that any two adjacent vertices have exactly 'a' common neighbours.
Let G_1 and G_2 be vertex-disjoint graphs. Then the join [3], $G_1 \vee G_2$, of G_1 and G_2 is the supergraph of the vertex disjoint union of G_1 and G_2 in which each vertex of G_1 is adjacent to every vertex of G_2.

All graph theoretic notations and terminology not mentioned here are from [3] and [6].

2. Inequalities relating the eigenvalues of $L(G), \Gamma(G)$ and $\Delta(G)$

Theorem 2.1 ([6]). *Let A, B be $n \times n$ Hermitian matrices. Then,*
$\lambda_i(A+B) \leq \lambda_j(A) + \lambda_{i-j+1}(B)$ $(n \geq i \geq j \geq 1)$
$\lambda_i(A+B) \geq \lambda_j(A) + \lambda_{i-j+n}(B)$ $(1 \leq i \leq j \leq n)$
where $\lambda_j(A)$ is the j-th eigenvalue of A.

We have the following observation from Theorem 2.1.

Observation

Consider a graph G with n vertices and m edges. Let $[L(G)], [\Gamma(G)]$ and $[\Delta(G)]$ be the adjacency matrices corresponding to $L(G), \Gamma(G)$ and $\Delta(G)$ respectively, then the matrices are of order $m \times m$ and satisfies the equations,
$\lambda_i[L(G)] \leq \lambda_j[\Gamma(G)] + \lambda_{i-j+1}[\Delta(G)]$ $(m \geq i \geq j \geq 1)$
$\lambda_i[L(G)] \geq \lambda_j[\Gamma(G)] + \lambda_{i-j+m}[\Delta(G)]$ $(1 \leq i \leq j \leq m)$

Note 1: The eigenvalues of $L(G) \geq -2$. But in the following Theorem we show that this case is not possible for Gallai graph and anti-Gallai graph.

Theorem 2.2. *In the class of Gallai graphs and anti-Gallai graphs, for any positive integer 'n'and negative integer '$-n$'there exist graphs with eigenvalue greater than 'n'and less than '$-n$'.*

Proof. $\Gamma(K^c_{1,(n+1)^2} \vee K_1)$ and $\Delta(K_{1,(n+1)^2} \vee K_1)$ contains $K_{1,(n+1)^2}$ as an induced subgraph. The eigenvalues of $K_{1,(n+1)^2}$ includes $-(n+1)$ and $(n+1)$. Hence for any positive integer 'n'and negative integer '$-n$'there exist graphs with eigenvalue greater than 'n'and less than '$-n$'. □

Note 2: For a graph G, the largest eigenvalue, $\lambda_1 = Sup_{||x||=1} X^\intercal AX$, and the smallest eigenvalue, $\lambda_n = Inf_{||x||=1} X^\intercal AX$, where $x \in R^n$ [6].

Theorem 2.3. *Let $\lambda_1, \lambda_1', \lambda_1''$ denote the largest eigenvalues of $L(G), \Gamma(G), \Delta(G)$ respectively and let $\lambda_m, \lambda_m', \lambda_m''$ denote the smallest eigenvalues of $L(G), \Gamma(G), \Delta(G)$ respectively, then,*

$$\lambda_1 \leq \lambda_1' + \lambda_1''.$$
$$\lambda_m \geq \lambda_m' + \lambda_m''.$$

Proof. From the above Note,

$$\begin{aligned}\lambda_1 &= Sup_{||x||=1} X^{\intercal}[L]X,\\ &= Sup_{||x||=1} X^{\intercal}([\Gamma]+[\Delta])X,\\ &\le Sup_{||x||=1} X^{\intercal}[\Gamma]X + Sup_{||x||=1} X^{\intercal}[\Delta]X,\\ &\le \lambda_1' + \lambda_1''.\end{aligned}$$

Also we have,

$$\begin{aligned}\lambda_m &= Inf_{||x||=1} X^{\intercal}[L]X,\\ &\ge Inf_{||x||=1} X^{\intercal}[\Gamma]X + Inf_{||x||=1} X^{\intercal}[\Delta]X,\\ &\ge \lambda_m' + \lambda_m''.\end{aligned}$$

□

3. Bounds for the eigenvalues of $\Gamma(G)$

Theorem 3.1. [14] *If G is $(K_4, diamond)$-free and if $\lambda_1, \lambda_2, ..., \lambda_n$ denote the eigenvalues of G, then the eigenvalues of $\Delta(G)$ are -1, 0 and 2 with multiplicities $\Sigma_{i=1}^{n} \frac{\lambda_i^3}{3}, \Sigma_{i=1}^{n}(\frac{\lambda_i^2}{2} - \frac{\lambda_i^3}{2})$ and $\Sigma_{i=1}^{n} \frac{\lambda_i^3}{6}$ respectively.*

Theorem 3.2. *Let $G = (n, k, a)$ be an edge regular graph then, $\rho(\Gamma(G)) = 2(k-a-1)$ and $\rho(\Delta(G)) = 2a$. Where ρ denotes the largest eigenvalue.*

Proof. Consider an arbitrary vertex x of $\Gamma(G)$. Let uv be the edge corresponding to this vertex in G. Then by the definition of $\Gamma(G)$,
$d_{\Gamma(G)}(x) = d_G(u) + d_G(v) - 2\text{a}{-}2$, where $d_{\Gamma(G)}(x)$ denotes the degree of x in $\Gamma(G)$ and $d_G(u), d_G(v)$ denote the degrees of u, v in G. Hence $\Gamma(G)$ is $2(k-a-1)$-regular and $\rho(\Gamma(G)) = 2(k-a-1)$.
Now consider an arbitrary vertex x in $\Delta(G)$. Let e be the corresponding edge in G. Then the degree of x in $\Delta(G)$ is twice the number of K_3 in which e belongs in G. Clearly it is $2a$. Therefore $\Delta(G)$ is $2a$-regular and $\rho(\Delta(G)) = 2a$. □

Note 3: If the eigenvalue λ appears t times in the spectrum, then we write it as λ^t.

Theorem 3.3. *Let $G = (n, k, 1)$ be an edge regular graph with eigenvalues $k, \lambda_2, ..., \lambda_n$. If $\lambda_1', \lambda_2', ..., \lambda_m'$ denote the eigenvalues of $\Gamma(G)$ then $\lambda_1' = 2k - 4, \lambda_i + k - 4 \le \lambda_i' \le \lambda_i + k - 1$ when $2 \le i \le n$ and $-4 \le \lambda_i' \le -1$, when $n + 1 \le i \le m$.*

Proof. Since the eigenvalues of G are $k, \lambda_2, ..., \lambda_n$, eigenvalues of $L(G)$ are $2k-2, \lambda_2+k-2, ..., \lambda_n+k-2, -2^{m-n}$. Given G is an edge regular graph with $a=1$. Therefore G is diamond-free and $\Delta(G)$ is the disjoint union of K_3's. It follows that the spectrum of $\Delta(G)$ contains only the eigenvalues $2, 0$ and -1 with different multiplicities and $\Gamma(G)$ is a $(2k-4)$- regular graph. Hence $\lambda'_1 = 2k-4$ and from the Observation, we have,

$\lambda_j[\Gamma(G)] \leq \lambda_i[L(G)] - \lambda_{i-j+m}[\Delta(G)]$ $(1 \leq i \leq j \leq n)$.

$\lambda_j[\Gamma(G) \geq \lambda_i[L(G)] - \lambda_{i-j+1}[\Delta(G)]$ $(n \geq i \geq j \geq 1)$.

Substitute $i=j$, in the above inequalities, we have,

$\lambda'_i \leq \lambda_i + k - 2 + 1$, when $2 \leq i \leq n$.

i.e., $\lambda'_i \leq \lambda_i + k - 1$, when $2 \leq i \leq n$.

And $\lambda'_i \geq \lambda_i + k - 2 - 2$, when $2 \leq i \leq n$.

i.e., $\lambda'_i \geq \lambda_i + k - 4$, when $2 \leq i \leq n$.

When $n+1 \leq i \leq m$, $\lambda'_i \leq -1$ and $\lambda'_i \geq -4$. □

We have the following theorem, proof of which is on similar line.

Theorem 3.4. *Let G be a k-regular graph with eigenvalues $k, \lambda_2, ..., \lambda_n$ and let $\lambda'_1, \lambda'_2, ..., \lambda'_m$ denote the eigenvalues of $\Gamma(G)$. If G is $(K_4, diamond)$-free then, $\lambda_i + k - 4 \leq \lambda'_i \leq \lambda_i + k - 1$ when $1 \leq i \leq n$ and $-4 \leq \lambda'_m \leq -1$, when $n+1 \leq i \leq m$.*

Theorem 3.5. [14] *In G, if two diamonds share at most one vertex then the spectrum of $\Delta(G)$ is $(\frac{1-\sqrt{17}}{2})^p, -1^a, 0^b, 1^p, 2^c, (\frac{1+\sqrt{17}}{2})^p$, where $p=$ the number of diamonds in G, $a = \Sigma_{i=1}^n \dfrac{\lambda_i^3}{3} + 2p, b = \Sigma_{i=1}^n (\dfrac{\lambda_i^2}{2} - \dfrac{\lambda_i^3}{2}) + p$, $c = \Sigma_{i=1}^n \dfrac{\lambda_i^3}{6} - 2p$ and $\lambda_1, \lambda_2, ..., \lambda_n$ denote the eigenvalues of G,.*

Theorem 3.6. *Let G be a k-regular graph with eigenvalues $k, \lambda_2, ..., \lambda_n$ and let $\lambda'_1, \lambda'_2, ..., \lambda'_m$ denote the eigenvalues of $\Gamma(G)$. In G, if two diamonds share at most one vertex, then, $\lambda_i + k - \frac{5+\sqrt{17}}{2} \leq \lambda'_i \leq \lambda_i + k - \frac{5-\sqrt{17}}{2}$, when $1 \leq i \leq n$ and $-\frac{5+\sqrt{17}}{2} \leq \lambda'_i \leq -\frac{5-\sqrt{17}}{2}$, when $n+1 \leq i \leq m$.*

Proof. As in the Theorem 3.3, here also the eigenvalues of $L(G)$ are $2k-2, \lambda_2+k-2, ..., \lambda_n+k-2, -2^{m-n}$. In G, since two diamonds share at most one vertex by Theorem 3.5, $\lambda''_m = \frac{1-\sqrt{17}}{2}$ and $\lambda''_1 = \frac{1+\sqrt{17}}{2}$, where λ''_1, λ''_m

denote the largest and least eigenvalues of $\Delta(G)$. Now substitute $i = j$, in the Observation, we have,

$\lambda_i' \leq \lambda_i + k - 2 - \frac{1-\sqrt{17}}{2}$, when $1 \leq i \leq n$

i.e., $\lambda_i' \leq \lambda_i + k - \frac{5-\sqrt{17}}{2}$, when $1 \leq i \leq n$

And $\lambda_i' \geq \lambda_i + k - 2 - \frac{1+\sqrt{17}}{2}$, when $1 \leq i \leq n$

i.e., $\lambda_i' \geq \lambda_i + k - \frac{5+\sqrt{17}}{2}$, when $1 \leq i \leq n$

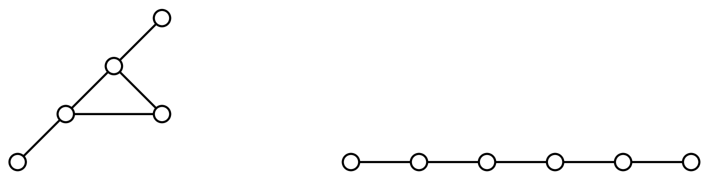
When $n+1 \leq i \leq m$, $\lambda_i' \leq -2 - \frac{1-\sqrt{17}}{2}$ and $\lambda_i' \geq -2 - \frac{1+\sqrt{17}}{2}$.

i.e., $-\frac{5+\sqrt{17}}{2} \leq \lambda_i' \leq -\frac{5-\sqrt{17}}{2}$, when $n+1 \leq i \leq m$. □

4. Characterization of $\Gamma(G)$ by spectra

In the class of Gallai graphs we mean by saying that a graph $\Gamma(G)$ is characterized by the spectrum if, $spec(\Gamma(G)) \cong spec(\Gamma(H))$ then $G \cong H$ [13]. Here P_n denotes the path on n vertices.

Result



The above two graphs are not isomorphic. But the Gallai graphs of the above graphs are isomorphic. Therefore $\Gamma(P_6)$ cannot be characterized by using the spectrum.

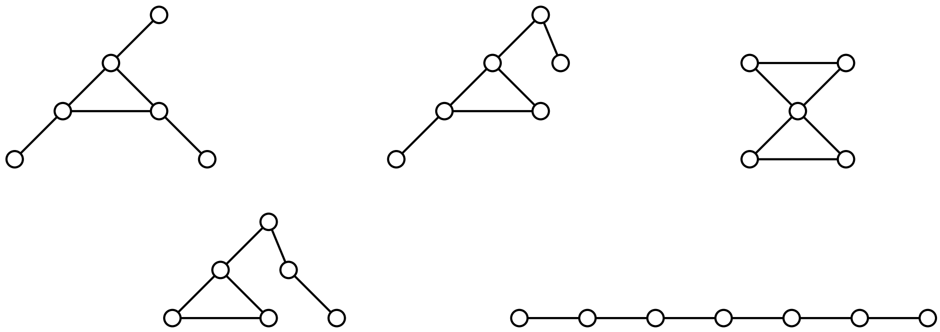
Theorem 4.1 ([7]). *The number of closed walks of length k in a graph G is equal to S_k, where*

$$S_k = \Sigma_{i=1}^{n} {\lambda_i}^k$$

Theorem 4.2. $\Gamma(P_7)$ *can be characterized by the spectrum in the class of Gallai graph of all graphs.*

Proof. Let H be a graph with the spectrum of $\Gamma(P_7)$, namely, -1.80194, $-1.24698, -.44504, .44504, 1.24698, 1.80194$. From Theorem 4.1, H is a K_3-free graph with 6 vertices and 5 edges. Also H is not a regular graph since,

$6\lambda_1 \neq \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_6^2$ [7]. Therefore H is not C_5. Since the largest eigenvalue of C_4 is $2 \nleq 1.80194$, C_4 is not an induced subgraph of H. That means H is a graph with 6 vertices and 5 edges with out C_3, C_4 and C_5 as induced subgraphs. It follows that H is a tree. When we consider the trees with 6 vertices only P_6 has the above spectrum. Hence $H \cong P_6$. Now we have to find the graph G with $\Gamma(G) \cong P_6$. Clearly G is a graph with 6 edges without C_4, C_5, C_6 and $K_{1,3}$ as induced subgraphs. Then G is one of the following 5 possible graphs.



Out of these 5 graphs only $\Gamma(P_7)) \cong P_6$. Hence $\Gamma(P_7)$ can be characterized by the spectrum, in the class of Gallai graph of all graphs. $\square$

Theorem 4.3. $\Gamma(K_{1,n})$ *can be characterized by the spectrum in the class of Gallai graph of all graphs.*

Proof. As in the above theorem, let H be a graph with the spectrum of $\Gamma(K_{1,n})$, namely, $-1^{n-1}, n-1$. From Theorem 4.1, H is a graph with n vertices and $\frac{n(n-1)}{2}$ edges. Also H is $(n-1)$-regular, since $n\lambda_1 = \lambda_1^2 + \lambda_2^2 + ... + \lambda_n^2$. Hence $H \cong K_n$. As in the above theorem when we consider G with $\Gamma(G) \cong K_n$, G is a graph with $n-1$ edges and any two edges of G are adjacent and not belong to a K_3. Hence $G \cong K_{1,n}$. $\square$

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A NOTE ON FRACTIONAL INEQUALITIES INVOLVING GENERALIZED KATUGAMPOLA FRACTIONAL INTEGRAL OPERATOR

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Abstract. The aim of this paper is to obtain some new fractional inequalities involving generalized Katugampola fractional integral operator by considering the extended Chebyshev functional in case of synchronous function. Also we obtain some fractional inequalities involving generalized Katugampola fractional integral.

1. Introduction

The integral inequalities, and in particular those related to fractional calculus operators, have considerable importance for the development of many branches of science and technology and its applications. In [6], the Chebyshev for two integrable functions f and g on $[a,b]$ is defined as

$$T(u,v)=\frac{1}{b-a}\int_a^b u(x)v(x)dx-\frac{1}{b-a}\left(\int_a^b u(x)dx\right)\frac{1}{b-a}\left(\int_a^b v(x)dx\right). \tag{1.1}$$

Many applications and several inequalities related to Chebyshev functional are found in [3, 4, 13, 14].
Let us consider the extended Chebyshev functional, [11]

$$\begin{aligned}T(u,v,p,q)=&\int_a^b q(x)dx\int_a^b p(x)u(x)v(x)dx+\int_a^b p(x)dx\int_a^b q(x)u(x)v(x)dx\\&-\left(\int_a^b p(x)u(x)dx\right)\left(\int_a^b q(x)v(x)dx\right)\\&-\left(\int_a^b q(x)u(x)dx\right)\left(\int_a^b p(x)v(x)dx\right),\end{aligned} \tag{1.2}$$

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where f and g are two integrable functions on $[a,b]$ and p and q are positive integrable functions on $[a,b]$. If f an g are synchronous on $[a,b]$, then $T(u,v,p,q) \geq 0$.
Nowadays, many mathematicians have established several fractional integral inequalities and its applications using the Hadamard, Riemann-Liouville, Erdélyi-Kober, Saigo, and the Katugampola fractional integral operators, see [2, 4, 7, 9, 10, 11, 15, 16, 18, 19, 21, 23, 27]. In [7, 8], the authors investigated fractional integral inequalities by considering Hadamard and generalized k-fractional integral operators for extended Chebyshev functional.
In [1, 20, 26, 28], the authors have obtained the Grüss-type, Chebyshev type and some other fractional inequalities for convex functions by employing generalized Katugampola fractional integrals. Motivated by the above work, the main objective of this article is to present some fractional inequalities using generalized Katugampola fractional integral operators for the extended Chebyshev functional.

2. Preliminaries

Here, we provide some basic definitions and remarks of generalized Katugampola fractional integrals which will be used in proving the main results, see [1, 16, 26, 28].

Definition 2.1. Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval in $\mathbb{R}$. The right and left side Riemann-Liouville fractional integrals of function ψ of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ are defined by, see [17, 22, 25]

$$(\mathcal{J}_{a+}^{\alpha}\psi)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\psi(t)}{(x-t)^{1-\alpha}} dt, \ (x > a), \tag{2.1}$$

and

$$(\mathcal{J}_{b-}^{\alpha}\psi)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\psi(t)}{(t-x)^{1-\alpha}} dt, \ (x < b). \tag{2.2}$$

Definition 2.2. The Liouville fractional integrals of a function ψ of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ are defined by, see [17, 22, 25]

$$(\mathcal{I}_{0+}^{\alpha}\psi)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\psi(t)}{(x-t)^{1-\alpha}} dt, \ (x \in \mathbb{R}^+). \tag{2.3}$$

Definition 2.3. Let (a,b) $(-\infty < a < b < \infty)$ be a finite or an infinite interval in $\mathbb{R}^+$. The right and left side Hadamard fractional integrals of real function $\psi \in L(a,b)$ of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ are defined by, see [17, 22, 25]

$$(\mathcal{H}^{\alpha}_{a+}\psi)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (\ln \frac{x}{t})^{\alpha-1} \frac{\psi(t)}{t} dt, \ (a < x < b), \tag{2.4}$$

and

$$(\mathcal{H}^{\alpha}_{b-}\psi)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (\ln \frac{t}{x})^{\alpha-1} \frac{\psi(t)}{t} dt, \ (a < x < b). \tag{2.5}$$

Definition 2.4. Let (a,b) $(-\infty \leq a < b \leq \infty)$ be a finite or an infinite interval on half axis $\mathbb{R}^+$. Also let $\Re(\alpha) > 0$ $\sigma \in \mathbb{R}^+$, $\eta \in \mathbb{C}$. The Erdélyi-Kober fractional integrals (left-side) of real function $\psi \in L(a,b)$ of order $\alpha \in \mathbb{C}$ are defined by, see [17, 19, 22, 24, 25]

$$(\mathcal{I}^{\alpha}_{a+,\sigma,\eta}\psi)(x) := \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma(\eta+1)-1}}{(x^{\sigma} - t^{\sigma})^{\alpha-1}} \psi(t) dt, \ (0 \leq a < x < b \leq \infty), \tag{2.6}$$

and

$$(\mathcal{I}^{\alpha}_{b-,\sigma,\eta}\psi)(x) := \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(\eta+1)-1}}{(t^{\sigma} - x^{\sigma})^{\alpha-1}} \psi(t) dt, \ (0 \leq a < x < b \leq \infty). \tag{2.7}$$

Definition 2.5. [5, 17] Consider the space $X^p_c(a,b)$ $(c \in \mathbb{R}, 1 \leq p \leq \infty)$, of those complex valued Lebesgue measurable functions ψ on (a,b) for which the norm $||\psi||_{X^p_c} < \infty$, such that

$$||\psi||_{X^p_c} = \left(\int_a^b |x^c \psi(x)|^p \frac{dx}{x} \right)^{\frac{1}{p}}, (1 \leq p < \infty)$$

and

$$||\psi||_{X^p_c} = esssup_{x \in (a,b)} [x^c |\psi|].$$

In particular, when $c = \frac{1}{p}$, the space $X^p_c(a,b)$ coincides with the space $L^p(a,b)$.

Definition 2.6. Let $[a,b] \subset \mathbb{R}$ be a finite interval. The Katugampola fractional integrals (left-side) of real function $\psi \in X^p_c(a,b)$ of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\rho \in \mathbb{R}^+$ are defined by, see [15, 19, 26]

$$({}^{\rho}\mathcal{I}^{\alpha}_{a+}\psi)(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^{\rho} - t^{\rho})^{1-\alpha}} \psi(t) dt, \ (x > a), \tag{2.8}$$

and

$$({}^{\rho}\mathcal{I}^{\alpha}_{b-}\psi)(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^{\rho} - x^{\rho})^{1-\alpha}} \psi(t) dt, \ (x < b). \tag{2.9}$$

Definition 2.7. Let $0 \leq a < x < b \leq \infty$. Also let $\psi \in X_c^p(a,b)$, $\alpha \in \mathbb{R}^+$, $\beta, \rho, \eta, k \in \mathbb{R}$. The generalized Katugampola fractional integrals (left and right sided) of a function ψ are defined, respectively, see [16, 26, 28, 29]

$$ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{a+;\eta,k}\psi(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\psi(\tau)d\tau,\ 0 \leq a < x < b \leq \infty, \tag{2.10}$$

and

$$ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{b-;\eta,k}\psi(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_x^b \frac{\tau^{k+\rho-1}}{(\tau^\rho-x^\rho)^{1-\alpha}}\psi(\tau)d\tau,\ 0 \leq a < x < b \leq \infty, \tag{2.11}$$

if the integral exist.

If we take $a = 0$, in (2.10), we have

$$ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}\psi(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\psi(\tau)d\tau. \tag{2.12}$$

Now, we define the following function as in [26, 28]: Let $x > 0, \alpha > 0, \rho, k, \beta, \eta \in \mathbb{R}$, then

$$ \Lambda^{\rho,\beta}_{x,k}(\alpha,\eta) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}d\tau = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)}\rho^{-\beta}x^{k+\rho(\eta+\alpha)}. \tag{2.13}$$

Remark 2.8. The fractional integral (2.10) contains five well-known fractional integrals as its particular cases, see [1, 15, 16, 20, 26, 28]

(1) Setting $k = 0, \eta = 0, a = 0$ and taking the limit $\rho \to 1$ in (2.10), the integral operator (2.10) reduces to the Riemann-Liouville fractional integral (2.3), see [17, 25].
(2) Setting $k = 0, \eta = 0$ and taking the limit $\rho \to 1$ in (2.10), the integral operator (2.10) reduces to the Liouville fractional integral (2.3), see [17, 25].
(3) Setting $\beta = \alpha, k = 0, \eta = 0$ and taking the limit $\rho \to 0^+$ with L' Hospital rule in (2.10), the integral operator (2.10) reduces to the Hadamard fractional integral (2.4) see [17, 25].
(4) Setting $\beta = 0, k = -\rho(\alpha+\eta)$ in (2.10), the integral operator (2.10) reduces to the Erdelyi-Kober fractional integral (2.7), see [24, 25, 19].
(5) Setting $\beta = \alpha, k = 0$ and $\eta = 0$ in (2.10), the integral operator (2.10) reduces to the Katugampola fractional integral (2.9), see [15].

Definition 2.9. Two functions u and v are called synchronous (asynchronous) functions on $[a,b]$ if

$$\Big(u(\tau)-u(\sigma)\Big)\Big(v(\tau)-v(\sigma)\Big) \geq \Big(\leq\Big)0,\ (\tau,\sigma\in[a,b]). \tag{2.14}$$

3. Main Results

Here, we propose some inequalities for extended Chebyshev functional in case of synchronous functions using generalized Katugampola fractional integral operator.

Lemma 3.1. *Let f and g be two integrable and synchronous functions on $[0,\infty)$. and $u,v:[0,\infty)\to[0,\infty)$. Then*

$$\begin{aligned}&{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[u(x)]\ \ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[vfg(x)]+\ \ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[v(x)]\ \ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[ufg(x)]\geq\\ &\quad {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[uf(x)]\ \ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[vg(x)]+\ \ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[vf(x)]\ \ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[ug(x)],\end{aligned} \tag{3.1}$$

hold, where for all $\alpha\geq 0$, $x>0$,$\beta,\rho,\eta,k\in\mathbb{R}$, we have

Proof: Since f and g are synchronous functions on $[0,\infty)$ for all $\tau\geq 0$, $\sigma\geq 0$, we have

$$(f(\tau)-f(\sigma))(g(\tau)-g(\sigma))\geq 0,(\tau,\sigma\in(0,\infty)). \tag{3.2}$$

From (3.2),

$$f(\tau)g(\tau)+f(\sigma)g(\sigma)\geq f(\tau)g(\sigma)+f(\sigma)g(\tau). \tag{3.3}$$

Now, multiplying both sides of equation (3.3) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}u(\tau),\tau\in(0,x),x>0$ which is positive, and integrating the obtained result with respect to τ from 0 to x, we get

$$\begin{aligned}&\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}u(\tau)f(\tau)g(\tau)d\tau\\ &+\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}u(\tau)f(\sigma)g(\sigma)d\tau\\ &\geq\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}x)u(\tau)f(\tau)g(\sigma)d\tau\\ &+\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}u(\tau)f(\sigma)g(\tau)d\tau,\end{aligned} \tag{3.4}$$

consequently,

$$\begin{aligned} &{}^{\sigma}\mathcal{J}_{\eta,k}^{\alpha,\beta}[ufg(x)] + f(\sigma)g(\sigma)^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[u(x)] \\ &\geq g(\sigma)^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[uf(x)] + f(\sigma)^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[ug(x)]. \end{aligned} \tag{3.5}$$

Multiplying both sides of (3.5) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}v(\sigma), \sigma \in (0,x), x > 0$ which is positive, and integrating the result with respect to σ from 0 to x, we get

$$\begin{aligned} &{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[ufg(x)]\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}v(\sigma)d\sigma \\ &+^{\rho} \mathcal{J}_{\eta,k}^{\alpha,\beta}[u(x)]\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}v(\sigma)f(\sigma)g(\sigma)d\sigma \\ &\geq^{\rho} \mathcal{J}_{\eta,k}^{\alpha,\beta}[uf(x)]\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}v(\sigma)g(\sigma)d\sigma \\ &+^{\rho} \mathcal{J}_{\eta,k}^{\alpha,\beta}[ug(x)]\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}v(\sigma)f(\sigma)d\sigma. \end{aligned} \tag{3.6}$$

which gives inequality (3.1).
Here, we propose our important result.

Theorem 3.2. *Let f and g be two integrable and synchronous functions on $[0,\infty)$, and $r,p,q : [0,\infty) \to [0,\infty)$. Then*

$$\begin{aligned} &2^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qfg(x)] +^{\rho} \mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pfg(x)]\right] + \\ &2^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rfg(x)] \geq \\ &{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pf(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qg(x)] +^{\rho} \mathcal{J}_{\eta,k}^{\alpha,\beta}[qf(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pg(x)]\right] + \\ &{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rf(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qg(x)] +^{\rho} \mathcal{J}_{\eta,k}^{\alpha,\beta}[qf(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rg(x)]\right] + \\ &{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rf(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pg(x)] +^{\rho} \mathcal{J}_{\eta,k}^{\alpha,\beta}[pf(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rg(x)]\right], \end{aligned} \tag{3.7}$$

hold, for all $\alpha \geq 0$, $x > 0$, $\beta, \rho, \eta, k \in \mathbb{R}$,

Proof: To prove Theorem 3.2, take $u = p$, $v = q$, and using Lemma 3.1, we get

$$\begin{aligned} &{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qfg(x)] +^{\rho} \mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pfg(x)] \geq \\ &{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pf(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qg(x)] +^{\rho} \mathcal{J}_{\eta,k}^{\alpha,\beta}[qf(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pg(x)]. \end{aligned} \tag{3.8}$$

Multiplying both sides of (3.8) by ${}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]$, we have

$$\begin{aligned}&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qfg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pfg(x)]\right]\geq\\&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pg(x)]\right].\end{aligned}\tag{3.9}$$

Again, take $u=r, v=q$, and using Lemma 3.1, we obtain

$$\begin{aligned}&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qfg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rfg(x)]\geq\\&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rg(x)].\end{aligned}\tag{3.10}$$

Multiplying both sides of (3.10) by ${}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]$, we have

$$\begin{aligned}&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qfg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rfg(x)]\right]\geq\\&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[qf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rg(x)]\right].\end{aligned}\tag{3.11}$$

With similar arguments as in equation (3.10) and (3.11), we can write

$$\begin{aligned}&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pfg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rfg](x)\right]\geq\\&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rg(x)]\right].\end{aligned}\tag{3.12}$$

Adding the inequalities (3.9), (3.11) and (3.12), we get inequality (3.7).

Lemma 3.3. *Let f and g be two integrable and synchronous functions on $[0,\infty)$, and $u,v:[0,\infty[\to[0,\infty)$, then*

$$\begin{aligned}&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[u(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[vfg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[v(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[ufg(x)]\geq\\&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[uf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[vg(x)]+{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[vf(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[ug(x)],\end{aligned}\tag{3.13}$$

hold, for all $x>0$, $k\geq 0$, $\alpha,\theta\geq 0$, $x>0$, $\beta,\pi,\rho,\eta,\in\mathbb{R}$.

Proof: Multiplying both sides of (3.5) by $v(\sigma)\frac{\rho^{1-\pi}t^{k}}{\Gamma(\theta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}$, $\sigma\in(0,t), t\in\mathbb{R}$ which (in view of the argument mentioned above in the proof of Theorem 3.1) remain positive. Then integrating resulting identity with respect to σ from 0 to x, we have

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[ufg(x)]\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}v(\sigma)d\sigma\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[u(x)]\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}v(\sigma)f(\sigma)g(\sigma)d\sigma\\
&\geq\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[uf(x)]\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}v(\sigma)g(\sigma)d\sigma\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[ug(x)]\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}v(\sigma)f(\sigma)d\sigma.
\end{aligned}\tag{3.14}$$

This gives the inequality (3.13).

Theorem 3.4. *Let f and g be two integrable and synchronous functions on $[0,\infty)$, and $r,p,q:[0,\infty)\to[0,\infty)$, then for all $k\geq 0$, $x>0$, $\alpha,\theta\geq 0$, $\beta,\pi,\rho,\eta,\in\mathbb{R}$, we have*

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)][\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[pfg(x)]+2\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}(qfg)(x)\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[q(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pfg(x)]]\\
&+\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[q(x)]+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[p(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]\right]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rfg(x)]\geq\\
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qg(x)]+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pg(x)]\right]+\\
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qg(x)]+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rg(x)]\right]+\\
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[q(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[pg(x)]+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[pf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[rg(x)]\right].
\end{aligned}\tag{3.15}$$

Proof: To prove Theorem 3.4, take $u=p$, $v=q$ and using Lemma 3.3, we get

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qfg(x)]+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[q(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pfg(x)]\geq\\
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qg(x)]+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pg(x)].
\end{aligned}\tag{3.16}$$

Now, multiplying both sides of (3.16) by ${}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]$, we have

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[p(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qfg(x)]+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[q(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pfg(x)]\right]\geq\\
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]\left[{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[pf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qg(x)]+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[qf(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[pg(x)]\right].
\end{aligned}\tag{3.17}$$

Taking $u = r,\ v = q$, and using Lemma 3.3, we obtain

$$\begin{aligned}&{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[r(x)]\,{}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[qfg(x)] + {}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[q(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[rfg(x)] \geq \\ &{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[rf(x)]\,{}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[qg(x)] + {}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[qf(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[rg(x)].\end{aligned} \tag{3.18}$$

Multiplying both sides by (3.18) ${}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[p(x)]$, we have

$$\begin{aligned}&{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[p(x)]\left[{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[r(x)]\,{}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[qfg(x)] + {}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[q(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[rfg(x)]\right] \geq \\ &{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[p(x)]\left[{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[rf(x)]\,{}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[qg(x)] + {}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[qf(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[rg(x)]\right].\end{aligned} \tag{3.19}$$

With similar arguments as in equation (3.18) and (3.19), we obtain

$$\begin{aligned}&{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[q(x)]\left[{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[r(x)]\,{}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[pfg(x)] + {}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[p(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[rfg(x)]\right] \geq \\ &{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[q(x)]\left[{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[rf(x)]\,{}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[pg(x)] + {}^{\rho}\mathcal{J}^{\theta,\pi}_{\eta,k}[pf(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[rg(x)]\right].\end{aligned} \tag{3.20}$$

Adding the inequalities (3.17), (3.19) and (3.20), we get the inequality (3.15).

Remark 3.5. If f, g, r, p and q satisfy the following conditions,

(1) The functions f and g is asynchronous on $[0, \infty)$.
(2) The functions r, p, q are negative on $[0, \infty)$.
(3) Two of the functions r, p, q are positive and the third is negative on $[0, \infty)$,

then the inequality (3.7) and (3.15) are reversed.

4. Some other fractional integral inequalities

Here, we establish fractional inequalities involving generalized Katugampola fractional integral.

Theorem 4.1. *Let f, g and r be three functions on $[0, \infty)$ such that*

$$(f(\tau) - f(\sigma))(g(\tau) - g(\sigma))(r(\tau) + r(\sigma)) \geq 0 \tag{4.1}$$

for all τ, σ. Then

$$\begin{aligned}&{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fgr(x)]\Lambda^{\rho,\beta}_{x,k}(\alpha,\eta) + {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fg(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[r(x)] \\ &\geq {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[g(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fr(x)] + {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[f(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[gr(x)],\end{aligned} \tag{4.2}$$

for all $k \geq 0$, $\alpha \geq 0$, $x > 0, \beta, \rho, \eta, \in \mathbb{R}$.

Proof:- From condition (4.1), for any τ, σ, we have

$$\begin{aligned} &f(\tau)g(\tau)r(\tau) + f(\tau)g(\tau)r(\sigma) + f(\sigma)g(\sigma)r(\tau) + f(\sigma)g(\sigma)r(\sigma) \\ &\geq f(\tau)g(\sigma)r(\tau) + f(\sigma)g(\tau)r(\tau) + f(\sigma)g(\tau)r(\sigma) + f(\tau)g(\sigma)r(\sigma). \end{aligned} \tag{4.3}$$

Multiplying both sides of equation (4.3) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}}, \tau \in (0, x), x > 0$ which is positive, and integrating the obtain result with respect to τ from 0 to x, we get

$$\begin{aligned} &\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau)g(\tau)r(\tau)d\tau \\ &+ \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau)g(\tau)r(\sigma)d\tau \\ &+ \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\sigma)g(\sigma)r(\tau)d\tau \\ &+ \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\sigma)g(\sigma)r(\sigma)d\tau \\ &\geq \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau)g(\sigma)r(\tau)d\tau \\ &+ \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\sigma)g(\tau)r(\tau)d\tau \\ &+ \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\sigma)g(\tau)r(\sigma)d\tau \\ &+ \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau)g(\sigma)r(\sigma)d\tau, \end{aligned} \tag{4.4}$$

which implies that

$$\begin{aligned} &{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fgr(x)] + r(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fg(x)] \\ &+\ f(\sigma)g(\sigma){}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)] + f(\sigma)g(\sigma)r(\sigma)\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \\ &\geq g(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fr(x)] + f(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[gr(x)] \\ &+ f(\sigma)r(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[g(x)] + g(\sigma)r(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[f(x)]. \end{aligned} \tag{4.5}$$

Multiplying both sides of equation (4.5) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho - \sigma^\rho)^{1-\alpha}}, \sigma \in (0, x), x > 0$ which is positive, and integrating the obtain result with respect to σ from 0 to x, we get

$$\begin{aligned}
&{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fgr(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}{\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})}}^{1-\alpha}d\sigma\\
&+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fg(x)]\,\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}r(\sigma)d\sigma\\
&+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[r(x)]\,\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}f(\sigma)g(\sigma)d\sigma\\
&+\Lambda^{\rho,\beta}_{x,k}(\alpha,\eta)\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}f(\sigma)g(\sigma)r(\sigma)d\sigma\\
&\geq{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fr(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}g(\sigma)d\sigma\\
&+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[gr(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}f(\sigma)d\sigma\\
&+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[g(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}f(\sigma)r(\sigma)d\sigma\\
&+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[f(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}g(\sigma)r(\sigma)d\sigma.
\end{aligned}\tag{4.6}$$

Hence

$$\begin{aligned}
&\Lambda^{\rho,\beta}_{x,k}(\alpha,\eta)\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fgr(x)]+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fg(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[r(x)]\\
&+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[r(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fg(x)]+\Lambda^{\rho,\beta}_{x,k}(\alpha,\eta)\,{}^{\rho}\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fgr(x)]\\
&\geq{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fr(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[g(x)]+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[gr(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[f(x)]\\
&+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[g(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fr(x)]+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[f(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[gr(x)].
\end{aligned}\tag{4.7}$$

This competes the proof of inequality (4.2).

Theorem 4.2. *Let f, g and r be three functions on $[0,\infty)$ such that*

$$(f(\tau)-f(\sigma))(g(\tau)+g(\sigma))(r(\tau)+r(\sigma))\geq 0\tag{4.8}$$

for all τ,σ. Then

$$\begin{aligned}
&{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[f(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[gr(x)]+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fr(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[g(x)]\\
&\geq{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[gr(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[f(x)]+{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[r(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[fg(x)],
\end{aligned}\tag{4.9}$$

for all $k\geq 0$,$\alpha\geq 0$, $x>0$,$\beta,\rho,\eta\in\mathbb{R}$.

Proof:- From condition (4.1), for any τ,σ, we have

$$\begin{aligned}
&f(\tau)g(\tau)r(\tau)+f(\tau)g(\tau)r(\sigma)+f(\tau)g(\sigma)r(\tau)+f(\tau)g(\sigma)r(\sigma)\\
&\geq f(\sigma)g(\tau)r(\tau)+f(\sigma)g(\tau)r(\sigma)+f(\sigma)g(\sigma)r(\tau)+f(\sigma)g(\sigma)r(\sigma).
\end{aligned}\tag{4.10}$$

Multiplying both sides of (4.10) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}, \tau\in(0,x), x>0$ which is positive, and integrating the obtain result with respect to τ from 0 to x, we get

$$\begin{aligned}
&\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} f(\tau)g(\tau)r(\tau)d\tau \\
&+ r(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} f(\tau)g(\tau)d\tau \\
&+ g(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} f(\tau)r(\tau)d\tau \\
&+ g(\sigma)r(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} f(\tau)d\tau \\
&\geq f(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} g(\tau)r(\tau)d\tau \\
&+ r(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} f(\tau)g(\tau)d\tau \\
&+ f(\sigma)g(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} r(\tau)d\tau \\
&+ f(\sigma)g(\sigma)r(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}} d\tau,
\end{aligned} \tag{4.11}$$

which implies that

$$\begin{aligned}
&{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fgr(x)] + r(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fg(x)] \\
&+\ g(\sigma){}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fr(x)] + g(\sigma)r(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[f(x)] \\
&\geq r(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[gr(x)] + r(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fg(x)] \\
&+ f(\sigma)g(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)] + f(\sigma)g(\sigma)r(\sigma)\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta).
\end{aligned} \tag{4.12}$$

With similar arguments as in inequality (4.6), we get

$$\begin{aligned}
&\Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fgr(x)] +\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fg(x)]\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)] \\
&+\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fr(x)]\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[g(x)] +\ {}^\rho\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[f(x)]\,{}^\rho\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[gr(x)] \\
&\geq\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[gr(x)]\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[f(x)] +\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fg(x)]\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)] \\
&+\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[r(x)]\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fg(x)] + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[fgr(x)].
\end{aligned} \tag{4.13}$$

This compete the proof of inequality (4.9).

5. Concluding Remarks

In this study, we introduced several fractional integral inequalities for extended Chebyshev functions by using generalized Katugampola fractional integral operators. Also, we investigated some other fractional integral inequalities by employing the given operators. The inequalities proposed in the present article are general than the existing inequalities. In particular, if we take, $k = 0, \eta = 0, a = 0$ and taking the limit as $\rho \to 1$ and $\rho \to 0^+$, then our results could be stated using the Riemann–Liouville and Hadamard fractional integrals respectively, see [7, 12]. If we take, $\beta = \alpha, k = 0$ and $\eta = 0$ then our results could be stated using the Katugampola fractional integral. The inequalities investigated in this paper give some contribution in the fields of fractional calculus and generalized Katugampola fractional integral operators.

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STANLEY–ELDER–FINE THEOREMS FOR COLORED PARTITIONS

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Abstract. We give a new proof of a partition theorem popularly known as Elder's theorem, but which is also credited to Stanley and Fine. We extend the theorem to the context of colored partitions (or prefabs). More specifically, we give analogous results for b-colored partitions, where each part occurs in b colors; for b-colored partitions with odd parts (or distinct parts); for partitions where the part k comes in k colors; and, overpartitions.

1. Introduction

The purpose of this paper is to extend a charming theorem in the theory of partitions which appeared in Stanley [9, Ch 1, Ex. 80], but is usually attributed to Elder, and more recently, has been found in the work of Fine; see Gilbert [8] for a comprehensive history. Our proof of this theorem uses recurrence relations; it extends naturally to encompass analogous results of Andrews and Merca [2] and Gilbert [8], and suggests their generalization to the context of colored partitions (or prefabs). In addition, we obtain analogous results for partitions where the kth part comes in k colors and for overpartitions.

We recall some of the terminology from the theory of integer partitions. A partition of n is a way of writing n as an unordered sum of numbers. It is represented as a sequence of non-increasing, non-negative integers

$$\lambda : \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots,$$

with

$$n = \lambda_1 + \lambda_2 + \lambda_3 + \cdots .$$

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The symbol $\lambda \vdash n$ is used to indicate that λ is a partition of n; we say λ has weight n. If $\lambda \vdash n$, we have

$$n = \sum_k k f_k$$

where the non-negative integers $f_k = f_k(\lambda)$ denote the frequency of k in λ, that is, the number of times k comes in λ. For example, the partition $4+3+3+2+1+1+1+1$ has frequencies: $f_1 = 4, f_2 = 1, f_3 = 2, f_4 = 1$.

One of the quantities in the Stanley–Elder–Fine theorem is

$$F_k(n) := \sum_{\lambda \vdash n} f_k(\lambda),$$

the total number of k's appearing in all the partitions of n. The other is

$$G_k(n) = \sum_{\lambda \vdash n} g_k(\lambda),$$

where

$g_k(\lambda) :=$ the number of parts which appear at-least k times in λ.

The Stanley–Elder–Fine theorem says that for all n,

$$F_k(n) = G_k(n), \tag{1.1}$$

for $k = 1, 2, \ldots, n$.

There are several proofs of this result, many of a combinatorial nature (see [8] for references). Here is another, rather simple, combinatorial proof of (1.1). We first show

$$F_k(n) = p(n-k) + p(n-2k) + p(n-3k) + \cdots, \tag{1.2}$$

for $k = 1, 2, \ldots$. (We take $p(m) = 0$ for $m < 0$.) Observe that for $k = 1, 2, \ldots, n$,

$$F_k(n) = p(n-k) + F_k(n-k),$$

because adding k to each partition of $n-k$ yields a partition of n, and vice-versa, on deletion of k from any partition containing k as a part, we obtain a partition of $n-k$. This gives (1.2) by iteration.

Next, consider $G_k(n)$. If we add $1+1+\cdots+1$ (k-times) to any partition of $n-k$, we obtain a partition of n where 1 appears as a part at-least k times; if we add $2+2+\cdots+2$ (k-times) to any partition of $n-2k$, we obtain a partition of n where 2 appears as a part at-least k times; and so on. The process is reversible. Thus

$$G_k(n) = p(n-k) + p(n-2k) + p(n-3k) + \cdots \tag{1.3}$$

for $k = 1, 2, 3, \ldots$; and $F_k(n) = G_k(n)$ for all $k = 1, 2, \ldots, n$.

The "counting by rows = counting by columns" quality of the Stanley–Elder–Fine theorem is reflected in this proof.

The objective of this paper is to study quantities given by expressions similar to (1.2) and obtain theorems analogous to (1.1). We consider colored partitions generated by

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^{b_k}},$$

where b_k is a sequence of non-negative numbers. These are called prefabs by Wilf [10, §3.14], but we prefer the imagery of partitions with colored parts. Each part k comes in b_k colors. They reduce to ordinary partitions when $b_k = 1$ for all k. We are able to extend (1.1) to the cases $b_k = b$ (a positive number) and when $b_k = k$ for all k. The analogues of the Stanley–Elder–Fine theorem given by Andrews and Merca [2] and Gilbert [8] extend to this context quite naturally.

In addition, we consider partition objects from generating functions that are products of such products; in particular, we consider overpartitions. An overpartition of n is a non-increasing sequence of natural numbers whose sum is n, where the first occurrence of a number may be overlined; the number of overpartitions of n is denoted by $\overline{p}(n)$.

We highlight one of our results on overpartitions. It is an analogue of Stanley's theorem, that is, (1.1) with $k = 1$.

Let $\overline{F}_1(n)$ be the number of times 1 or $\overline{1}$ appears in the overpartitions of n. Let $\overline{G}_k(n)$ count the number of parts (overlined or not) repeated at-least k times in an overpartition, summed over all overpartitions of n. For example, the overpartition $3 + 2 + 2 + \overline{2} + 1 + \overline{1}$ has the part 2 repeated three times; this part contributes 1 to $\overline{G}_k(11)$ for $k = 1, 2, 3$. An extension of Stanley's theorem is as follows.

Theorem 1.1. *The number of times* 1 *or* $\overline{1}$ *appears as a part in the overpartitions of* n *is equal to the difference of the number of times a part (ordinary or overlined) appears at-least once with the number of times a part appears at-least thrice in the overpartitions of* n*. That is, for* $n = 1, 2, \ldots,$

$$\overline{F}_1(n) = \overline{G}_1(n) - \overline{G}_3(n).$$

The rest of the paper is organized as follows. In §2, we prove an analogue of (1.2) for colored partitions. In §3 we consider b-colored partitions, where each part has b colors. Next, in §4, we consider b-colored partitions with odd or distinct parts. In §5 we consider partitions where the part k comes in k colors. The number of such partitions is the same as the number of plane partitions of n. Next, in §6 we consider overpartitions; the proof of Theorem 1.1 appears here. We conclude in §7 by giving credit where credit is due.

2. The frequency function for colored partitions

The objective of this section is to obtain a key relation for the frequency function for colored partitions. We use notation from [5] to represent colored partitions.

Let U_k represent a set containing b_k copies of k. The elements of U_k are represented as $k_1, k_2, \ldots$. The elements of U_k can be regarded as k with different colors. Consider two sets U_j and U_k, containing, respectively, b_j and b_k colors. We now use the symbol $U_j + U_k$ to denote the set of partitions

$$U_j + U_k := \{j_a + k_b : j_a \in U_j, k_b \in U_k\}.$$

Here $2U_j$ represents $U_j + U_j$. This definition is extended by induction to finite sums $\sum a_i U_i$ where $a_i \geq 0$. By a colored partition of weight n, we mean an element π where

$$\pi \in \sum_i a_i U_i,$$

with $|\pi| = \sum_{i=1}^{n} i a_i = n$.

For convenience, we use the symbols U_k even if the relevant sets have one element. In this case, we can say $\pi = \sum_k f_k U_k$ represents the partition $\sum_k k f_k$ in which the frequency of k is f_k.

For a partition π, let $f_{k_c}(\pi)$ be the number of parts equal to k_c in π, i.e., the frequency of k_c in π. Then the frequency of k in π is

$$f_k(\pi) = \sum_{k_c \in U_k} f_{k_c}(\pi).$$

We denote the sum of the frequencies of all partitions of size n by $F_k(n)$.

Theorem 2.1. *Let $h(n)$ represent the number of colored partitions of size n where k comes in b_k colors. Let $F_k(n)$ be the frequency of k in all the* *partitions of n. Then we have the recurrence relation*

$$F_k(n) = b_k \left(h(n-k) + h(n-2k) + h(n-3k) + \cdots\right) \tag{2.1}$$

Remark 2.2. Here b_k is a sequence of non-negative integers. A more general version of this theorem, where the b_k are complex numbers, is proved in [5].

Proof. We prove

$$F_k(n) = F_k(n-k) + b_k h(n-k). \tag{2.2}$$

The proof is an easy extension of the argument when $b_k = 1$ for all k. If we delete a part k_c in a colored partition of n containing that part, we obtain a partition of $n-k$. Vice versa, if we add a part k_c to a partition of $n-k$, we obtain a partition of n. Since there are b_k choices for k_c, the right hand side of (2.2) counts the total number of k's in any color appearing in colored partitions of n, and thus it equals the left hand side. □

3. Partitions with the same number of colors for each part

We consider b-colored partitions where each part k comes in b colors, where b is a positive integer. A Stanley–Elder–Fine theorem is as easy to obtain in this context as the $b = 1$ case. Let $h(n)$ represent the number of b-colored partitions of n. Let $k_1, k_2, \ldots, k_b$ represent the colored parts. Let $F_k(n)$ be the frequency of k in all the partitions of n. From (2.1), we have

$$F_k(n) = bh(n-k) + bh(n-2k) + bh(n-3k) + \cdots . \tag{3.1}$$

Let π be a partition. As before, let

$$g_k(\pi) := \text{the number of parts which appear at-least } k \text{ times in } \pi,$$

and

$$G_k(n) = \sum_{|\pi|=n} g_k(\pi).$$

Thus $G_k(n)$ is the number of times a part appears at-least k times in a partition, summed over all the partitions of n. Then we have

Theorem 3.1. *Let $F_k(n)$ and $G_k(n)$ be as above. Then, for all $n = 1, 2, \ldots,$*

$$F_k(n) = G_k(n),$$

for $k = 1, 2, \ldots, n$.

Proof. The proof is virtually the same as the $b = 1$ case. Note that if we add a r (of any color) to any partition of $n - r$, we obtain a partition of n which has at-least one r as a part. We can add any one of the b r's. This can be done for each $r = 1, 2, 3, \ldots$. Thus

$$G_1(n) = bh(n-1) + bh(n-2) + \cdots + bh(0),$$

since every part from every partition of n which is repeated at-least once will be accounted for (uniquely) in this way.

In general, we see that if we add $1_c + 1_c + \cdots + 1_c$ (k-times) to any partition of $n - k$, we obtain a partition of n where 1_c appears as a part at-least k times; if we add $2_c + 2_c + \cdots + 2_c$ (k-times) to any partition of $n - 2k$, we obtain a partition of n where 2_c appears as a part at-least k times; and so on. Thus

$$G_k(n) = bh(n-k) + bh(n-2k) + bh(n-3k) + \cdots$$

for $k = 1, 2, 3, \ldots$. In view of (3.1), $G_k(n)$ and $F_k(n)$ are equal. □

Next, we consider the quantity $H_k(n)$, defined as the sum of parts divisible by k, counted without multiplicity, in all the b-colored partitions of n. As an example, consider an ordinary partition (that is, $b = 1$) represented by $2U_3 + 4U_6$ or $3+3+6+6+6+6$. This contributes $3+6 = 9$ to the sum. On the other hand, a 2-colored partition $3_1 + 3_2 + 6_1 + 6_1 + 6_1 + 6_2 \in 2U_3 + 4U_6$ contributes $3 + 3 + 6 + 6 = 18$ to $H_3(30)$.

Theorem 3.2. *Let $F_k(n)$ and $H_k(n)$ be as above. Then for all n, we have*

$$kF_k(n) = H_k(n) - H_k(n-k).$$

Remark 3.3. When the number of colors $b = 1$, i.e., in the case of ordinary partitions, this result reduces to a result of Andrews and Merca [2].

Proof. We first show

$$H_k(n) = bkh(n-k) + 2bkh(n-2k) + 3bkh(n-3k) + \cdots. \tag{3.2}$$

The argument is similar to the one for $G_k(n)$. If we add $(rk)_c$ (the part rk in color c) to any partition of $n - rk$, we obtain a partition with $(rk)_c$ as a part. This contributes rk to $H_k(n)$. Conversely, if we delete $(rk)_c$ in a partition of n where it comes as a part, we obtain a partition of $n - rk$. This shows (3.2).

Now it is clear that $H_k(n) - H_k(n-k)$ equals $kF_k(n)$ by (3.1). □

4. Partitions with odd and distinct parts

We consider b-colored partitions with all parts odd (which all come in b-colors). Let $h(n)$ now denote the number of b-colored partitions with only odd parts. An easy extension of Euler's ODD=DISTINCT theorem (see [4, eq. (2.1)]) says that $h(n)$ is also the number of b-colored partitions with distinct parts. Here the $h(n)$ are generated by

$$\prod_{k=1}^{\infty}\left(1+q^k\right)^b = \prod_{k=1}^{\infty}\frac{1}{\left(1-q^{2k-1}\right)^b}.$$

Let $F_k^o(n)$ denote the corresponding frequency function. Then we have

$$F_k^o(n) = \begin{cases} F^o(n-k)+bh(n-k), & \text{if } k \text{ is odd;}\\ 0, & \text{if } k \text{ is even.}\end{cases} \tag{4.1a}$$

$$= \begin{cases} h(n-k)+h(n-2k)+h(n-3k)+\cdots, & \text{if } k \text{ is odd;}\\ 0, & \text{if } k \text{ is even.}\end{cases} \tag{4.1b}$$

Let $G_k^o(n)$ be the number of times a part appears at-least k times in a partition, summed over all the b-colored partitions of n with odd parts.

Theorem 4.1. *Let $F_k^o(n)$ and $G_k^o(n)$ be as above and let k be an odd number. Then, for all $n = 1, 2, \ldots,$*

$$F_k^o(n) = G_k^o(n) + G_k^o(n-k),$$

for $k = 1, 3, 5, \ldots.$

Remark 4.2. In the case of ordinary partitions, where $b = 1$, this theorem reduces to an observation of Gilbert [8, Th. 8].

Proof. If we add $1_c + 1_c + \cdots + 1_c$ (k-times) to any partition of $n-k$, we obtain a partition of n where 1_c appears as a part at-least k times; if we add $3_c + 3_c + \cdots + 3_c$ (k-times) to any partition of $n-3k$, we obtain a partition of n where 3_c appears as a part at-least k times; and so on. Thus

$$G_k^o(n) = bh(n-k) + bh(n-3k) + bh(n-5k) + \cdots \tag{4.2}$$

for $k = 1, 2, 3, \ldots.$ Note that this applies even if k is not an odd number.

When k is odd, we see from (4.1b) that

$$G_k^o + G_k^o(n-k) = F_k^o(n),$$

This proves the theorem. $\square$

Next let $F^d_k(n)$ denote the frequency of k in b-colored partitions with all parts distinct. The differently colored parts of the same weight are considered distinct. For example, $3_1 + 3_2 + 5_2$ is considered a 2-partition of 11 with distinct parts. This partition contributes 2 to $F^d_3(11)$. It is easy to see that

$$F^d_k(n) = bh(n-k) - F^d_k(n-k) \tag{4.3a}$$
$$= bh(n-k) - bh(n-2k) + bh(n-3k) - bh(n-4k) + \cdots . \tag{4.3b}$$

To obtain (4.3a), observe that a b-colored partition of n (with distinct parts) which contains k_c as a part is obtained by adding k_c to any partition of $n-k$ which does not have k_c as a part. So the number of distinct partitions of n containing k_c is $h(n-k) - F_{k_c}(n)$, where $F_{k_c}(n)$ is number of k_c's in the distinct partitions of $n-k$ (and also the number of distinct partitions containing k_c as a part). Now summing over all colors, we obtain (4.3a).

Since the number $h(n)$ of b-colored partitions with odd parts and distinct parts are the same, equation (4.3b), along with (4.2) immediately yields the following theorem.

Theorem 4.3. *Let $F^d_k(n)$ and $G^o_k(n)$ be as defined above. Then*

$$F^d_k(n) = G^o_k(n) - G^o_k(n-k),$$

for $k = 1, 2, 3, \ldots$.

Remark 4.4. When $b = 1$, Theorem 4.3 reduces to Gilbert [8, Th. 9].

5. Partitions with k copies of k

Next we consider a special case of colored partitions generated by the product

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}.$$

We reuse the notation $h(n)$ to denote the number of such partitions of n. The notations for $F_k(n)$ and $G_k(n)$ are also reused.

Theorem 5.1. *Consider the set of partitions of n where each part k comes in k colors. Let $F_k(n)$ be the number of k's (of any color) appearing in all such partitions of n. Let $G_k(n)$ denote the number of parts that appear at-least k times in such a partition, summed over all such partitions. Then,* *for all* $n = 1, 2, \ldots,$

$$F_k(n) = k\big(G_k(n) - G_k(n-k)\big),$$

for $k = 1, 2, \ldots, n.$

Proof. Observe that

$$G_k(n) = 1 \cdot h(n-k) + 2 \cdot h(n-2k) + 3 \cdot h(n-3k) + \cdots$$

for $k = 1, 2, 3, \ldots$.

Thus we have

$$G_k(n) - G_k(n-k) = h(n-k) + h(n-2k) + \cdots,$$

and, by Theorem 2.1,

$$\begin{aligned} F_k(n) &= k\big(h(n-k) + h(n-2k) + h(n-3k) + \cdots\big) \\ &= k\big(G_k(n) - G_k(n-k)\big). \end{aligned}$$

□

6. Overpartitions

Overpartitions can be represented as partitions in two symbols U and V. The partitions represented by U are ordinary partitions and the partitions represented by V are distinct. In a sense (to be explained shortly), overpartitions are convolutions of these two types of partitions. One of our theorems in this section shows how one can simply put together the respective results for two partition functions to obtain a new theorem for their convolution. In addition, we prove Theorem 1.1. Its proof is more intricate than what we have encountered so far.

The overpartitions of n can be formed by adding partitions of k in U with a distinct partition of $n-k$ in V. For example, here is a way to list the overpartitions of 4. First we list ordinary partitions up to 4 and then add them with partitions with distinct parts written in reverse order. (As before, we use the symbols U and V even if the relevant sets have one element.) Thus, for example, in case U_j has just one element, the symbol U_j represents j. In Table 1 we have listed the partitions of $m \leq 4$ in U and partitions with distinct parts in reverse order in V. For example, $2U_1 + V_2$ represents $\overline{2} + 1 + 1$ and $U_2 + V_2$ represents $\overline{2} + 2$.

m	Partitions of m	Partitions of $n-m$ into distinct parts	Partitions of $n-m$ into odd parts
0	-	V_1+V_3, V_4	$4W_1, W_1+W_3$
1	U_1	V_1+V_2, V_3	$3W_1, W_3$
2	$2U_1, U_2$	V_2	$2W_1$
3	$3U_1, U_1+U_2, U_3$	V_1	W_1
4	$4U_1, 2U_1+U_2, U_1+U_3, 2U_2, U_4$	-	-

TABLE 1. Overpartitions and odd-overlined partitions for $n=4$

Evidently,

$$\sum_m p(m)p(n-m \mid \text{distinct parts}) = \overline{p}(n),$$

a convolution of two sums; thus the generating function of overpartitions is the product of the respective generating functions:

$$\overline{Q}(q) = \sum_{n\geq 0} \overline{p}(n)q^n = \prod_{k=1}^{\infty} \frac{1+q^k}{1-q^k}.$$

It is in this sense we describe overpartitions as convolutions of ordinary partitions with partitions of distinct parts.

Since the number of partitions into distinct parts equals the number of odd partitions, it is natural to consider the partitions formed by adding an ordinary partition of m in u with a partitions of $n-m$ with odd parts in W (see Table 1). These are equinumerous to overpartitions. We call them *odd-overlined* partitions.

Consider colored overlined partitions (of both kinds) generated by the generating functions

$$\sum_{n=0}^{\infty} h(n)q^n = \prod_{k=1}^{\infty} \frac{\left(1+q^k\right)^s}{\left(1-q^k\right)^r} = \prod_{k=1}^{\infty} \frac{1}{\left(1-q^k\right)^r\left(1-q^{2k-1}\right)^s}.$$

Here the ordinary parts are r-colored and the distinct/odd parts are colored in s colors.

We mix and match the notations of §3 and §4. So, for example, $h(n)$ will refer to the number of overpartitions (respectively, odd-overlined partitions), $F_k(n)$ refers to the frequency of k in appearing in ordinary partition (in r colors) contained in the overline partition, and $F_k^d(n)$ and $F_k^o(n)$ are the frequencies of k of the overlined parts which come in s colors. Similarly, let $G_k(n)$, $G_k^o(n)$ be defined as earlier. Then we have:

Theorem 6.1. *Let $F_k(n)$, $F_k^o(n)$, $F_k^d(n)$, $G_k(n)$, and $G_k^o(n)$ be as defined above, in the context of colored overpartitions/odd-overlined partitions. Then, for all $n = 1, 2, \ldots,$*

$$\begin{aligned} F_k(n) &= G_k(n), \text{ for } k = 1, 2, 3, \ldots; \\ F_k^d(n) &= G_k^o(n) - G_k^o(n-k), \text{ for } k = 1, 2, 3, \ldots; \\ F_k^o(n) &= G_k^o(n) + G_k^o(n-k), \text{ for } k = 1, 3, 5, \ldots. \end{aligned}$$

Remark 6.2. One can get analogous theorems for partitions where each part k comes in $k+b$ colors, by combining the considerations of §3 with §5. Note that the colored partitions considered in §3 and §4 can be considered as convolutions too.

Before concluding, we prove Theorem 1.1 which is of a different nature than those studied above. Consider overpartitions with all parts of a single color generated by $\overline{Q}(q)$. For this theorem, we prefer the imagery of overpartitions where the part in V is overlined.

Let $F_k(n)$ and $F_d(k)$ be as above. Let

$$\overline{F}_k(n) = F_k(n) + F_k^d(n).$$

Thus, $\overline{F}_1(n)$ is the number of overpartitions of n with 1 or $\overline{1}$ as parts. Further, recall that $\overline{G}_k(n)$ counts the number of parts repeated at-least k times in an overpartition, summed over all overpartitions of n.

Proof of Theorem 1.1. It is easy to see (by the arguments used to obtain (2.1) and (4.3b)) that

$$F_1(n) = \overline{p}(n-1) + \overline{p}(n-2) + \overline{p}(n-2) + \overline{p}(n-3) + \cdots,$$

and

$$F_1^d(n) = \overline{p}(n-1) - \overline{p}(n-2) + \overline{p}(n-2) - \overline{p}(n-3) + \cdots,$$

so

$$\overline{F}_1(n) = 2\big(\overline{p}(n-1) + \overline{p}(n-3) + \overline{p}(n-5) + \cdots\big). \tag{6.1}$$

To obtain an expression for the right hand side, we need the following ancillary counting functions. Let $\overline{p}_m(n)$ be the number of overlined partitions which have m as a part. Note that

$$\overline{p}_m(n) = \overline{p}(n-m). \tag{6.2}$$

We also need the following functions:

$$\begin{aligned}
\overline{O}_m(n) &:= \text{the number of overpartitions where the part } m \text{ or } \overline{m} \text{ appears at-least } \textit{once};\\
O_m(n) &:= \text{the number of overpartitions where the part } m \text{ appears at-least once};\\
O_{\overline{m}}(n) &:= \text{the number of overpartitions where the part } m \text{ does not appear and } \overline{m} \text{ appears};\\
\overline{T}_m(n) &:= \text{the number of overpartitions where the part } m \text{ or } \overline{m} \text{ appears at-least } \textit{thrice}.
\end{aligned}$$

Evidently

$$\overline{O}_m(n) = O_m(n) + O_{\overline{m}}(n);$$

$$\sum_{m=1}^{n} \overline{O}_m(n) = \overline{G}_1(n);$$

and,

$$\sum_{m=1}^{n} \overline{T}_m(n) = \overline{G}_3(n).$$

To prove the theorem, we find expressions for $\overline{O}_m(n)$ and $\overline{T}_m(n)$ in terms of $\overline{p}(n)$.

Note that

$$\begin{aligned}
O_1(n) &= p(n-1)\\
O_{\overline{1}}(n) &= \overline{p}(n-1) - \overline{p}_1(n-1) - O_{\overline{1}}(n-1)\\
&= \overline{p}(n-1) - 2\overline{p}(n-2) + 2\overline{p}(n-3) - \overline{p}(n-4) + \cdots .
\end{aligned}$$

The first of these follows because we obtain an overpartition of n with 1 as a part by adding a 1 to each overpartition of $n-1$. For $O_{\overline{1}}(n)$ we note that we can add a $\overline{1}$ to each overpartition of $n-1$, which has neither 1 nor $\overline{1}$ as a part. Finally, the last line follows from (6.2) and iteration.

From the above, we find that

$$\overline{O}_1((n) = 2\left(\overline{p}(n-1) - \overline{p}(n-2) + \overline{p}(n-3) - \overline{p}(n-4) + \cdots\right).$$

Similarly, we have, for $m = 1, 2, \dots, n$

$$\overline{O}_m((n) = 2\,(\overline{p}(n-m) - \overline{p}(n-2m) + \overline{p}(n-3m) - \overline{p}(n-4m) + \cdots)\,. \tag{6.3}$$

Next, we note that

$$\begin{aligned}\overline{T}_1(n) &= \overline{p}(n-3) + O_{\overline{1}}(n-2)\\ &= 2\,(\overline{p}(n-3) - \overline{p}(n-4) + \overline{p}(n-5) - \overline{p}(n-7) + \cdots)\end{aligned}$$

The first of these is true because any overpartition where 1 comes at-least three times is obtained by adding $1+1+1$ to an overpartition of $n-3$ or by adding $1+1$ to an overpartition of $n-2$ which does not have a 1 but has an $\overline{1}$. The second follows by using the formula for $O_{\overline{1}}(n)$ computed above.

Similarly, for $m = 1, 2, \dots, n$,

$$\overline{T}_m(n) = 2\,(\overline{p}(n-3m) - \overline{p}(n-4m) + \overline{p}(n-5m) - \overline{p}(n-6m) + \cdots) \tag{6.4}$$

Finally, we have

$$\begin{aligned}\overline{G}_1(n) - \overline{G}_3(n) &= \sum_{m=1}^{n} \overline{O}_m(n) - \overline{T}_m(n)\\ &= \sum_{m=1}^{n} 2\big(\overline{p}(n-m) - \overline{p}(n-2m)\big) \text{ (using (6.3) and (6.4))}\\ &= 2\big(\overline{p}(n-1) + \overline{p}(n-3) + \overline{p}(n-5) + \cdots\big)\\ &= \overline{F}_1(n),\end{aligned}$$

using (6.1). This completes the proof. □

7. Closing credits

The key idea in our extensions of the Stanley–Elder–Fine theorem is (2.1). Given this, it is easy to manipulate the expression for $G_k(n)$ corresponding to the choice of b_k. Even so, the corresponding theorems of Andrews and Merca [2] and Gilbert [8] have motivated the form of our theorems in §3 and §4. In particular, the definition of $H_k(n)$ for ordinary partitions in [2] was very useful.

As we saw in §6, we can mix and match to find Stanley–Elder–Fine theorems for partitions obtained from the convolution of two different kinds of partitions. In addition to overpartitions, many such partition functions have appeared in the literature, and this technique can be used to give such theorems of them.

We mention some related work. Banerjee and Dastidar [6] gave a combinatorial proof too, but it is much more intricate than the one given here. Their colored partitions are different from ours; they are closer in spirit to the work in §6. Dastikar and Sen Gupta [7] have given an extension of Stanley's theorem which comes from summing (1.3) for $k = 1, 2, \ldots, k$ and noting that the sum equals $F_1(n)$. Their results can be immediately extended (as in this paper) to the context of colored partitions. Andrews and Deutsch [3] have a different generalization of Elder's theorem. Their starting point and key argument is not far from ours, but they have taken a different path to generalization; see also [1].

The relations (2.2) and (2.1) have number-theoretic consequences. These are studied by the authors in [5].

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POLYMATH 14: GROUPS WITH NORMS*

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Abstract. This note describes a recent Polymath project, which provided novel characterizations – from analysis and geometry – of a basic algebraic notion: abelian torsion-free groups. In addition to the mathematical novelty of such a result, the process of discovery was also uncommon for mathematics: through a crowdsourced attack, including assistance from a computer, and with all progress recorded in realtime on the blog of Terence Tao (UCLA). This work formed the basis of the 2022 Hansraj Gupta Memorial Award Lecture, at BIT Mesra in the annual meeting of the Indian Mathematical Society.

1. Norms on groups

One of the first things a mathematics student sees in college is the notion of a group – e.g., abelian groups. This is the start of algebra, but of course, a group structure underlies much of analysis, geometry, and probability. The goal in this note is to state and prove some novel characterizations of abelian torsion-free groups using geometry and analysis, which were motivated by probability. Additional details can be found in the full paper [5].

We begin with the aforementioned probabilistic motivation. In mathematics, one often tries to prove results in as great generality as they can be stated in – i.e., by weakening and removing "extra" hypotheses that are not required. For example, the statement "$1 - 1 = 0$" seemingly requires one to work in the reals $\mathbb{R}$ or maybe the integers $\mathbb{Z}$, but upon closer inspection it holds in *any group*: $g \cdot g^{-1} = e$. Similar examples can be found in other areas – for example, Pythagoras's theorem holds not only in the Euclidean plane $\mathbb{R}^2$ but in any inner product space.

It is in this context that we begin. In prior joint work [2], we had shown that a fundamental inequality in probability theory (attributed to

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Hoffmann-Jørgensen) holds for random variables that take values not merely in $\mathbb{R}$, nor even in some Banach space, but even more generally: in any topological semigroup $\mathscr{G}$ with a metric on it. Thus, $\mathscr{G}$ has an associative binary operation, but this need not be commutative, nor do we require an identity (which is needed in a **monoid**), let alone an inverse function.

Then in [3], we showed another probability inequality – attributed to Khinchin and Kahane – again not just over $\mathbb{R}$, but over any topological abelian group which possesses a stronger form of a metric: a norm.

Definition 1.1. A *length function* – on a monoid $(G, \cdot, e)$ is a map $\|\cdot\| : G \to \mathbb{R}_{\geqslant 0}$ which satisfies the properties of a metric:

(1) Positivity: $\|g\| > 0$ if $g \neq e$, and $\|e\| = 0$,
(2) Triangle inequality/sub-additivity: $\|gh\| \leqslant \|g\| + \|h\|$,

for all $g, h \in G$. If moreover G is a group, then we also require the symmetry property: $\|g\| = \|g^{-1}\|$ for all $g \in G$.

For G a monoid or group, if a length function $\|\cdot\|$ also satisfies the norm-property/homogeneity: $\|g^n\| = |n| \cdot \|g\|$ for $g \in G$ and n any integer, then we call $\|\cdot\|$ a *norm* on G.

The norm-property is modeled after the same property in a normed linear space $\mathbb{B} : \|c\mathbf{v}\| = |c| \cdot \|\mathbf{v}\|\ \forall c \in \mathbb{R}, \mathbf{v} \in \mathbb{B}$; of course, in a general group G one can only take integer powers. (We use powers g^n instead of the additive notation $c\mathbf{v}$ since G need not be abelian.)

Having defined a norm on G, what are examples of such? The natural motivating example is the one coming from functional analysis, i.e. Banach spaces, or normed linear spaces. More generally, any additive subgroup $G \leqslant (\mathbb{B}, +, \|\cdot\|)$ of a normed linear space is a(n abelian) group with a norm. The punchline of this work – i.e. of [5] – is that **there are no others**!

The goal of this article is to state this fact more precisely as a characterization result – along with other equivalent conditions – and to explain the proof. We begin with some preliminary observations. It is clear that if $\|\cdot\|$ is a norm on a monoid (e.g. a group) G, then G is *torsion-free* – i.e., if $g \neq e$ in G and $n > 0$ then $g^n \neq e$. This is because $\|g^n\| = |n| \cdot \|g\| > 0$. This already shows that no finite group can have a norm on it.

The next observation is that every norm is *conjugation-invariant* (a fact shown in [5] during the course of proving the main result below):

$$\|ghg^{-1}\| = \|g\|, \quad \forall g, h \in G \tag{1.1}$$

for any group G with a norm. Indeed, by the triangle inequality,

$$n\|ghg^{-1}\| = \|(ghg^{-1})^n\| = \|gh^n g^{-1}\| \leqslant \|g\| + \|g^{-1}\| + n\|h\|$$

for all integers $n > 0$. Now divide by n and let $n \to \infty$; by the Archimedean property, we see that $\|ghg^{-1}\| \leqslant \|h\|$, and (1.1) follows from the symmetry of conjugation. Notice also that length functions are in one-to-one correspondence with left-invariant metrics d on G:

$$\|g\| \mapsto d(e, g), \qquad d(g, h) \mapsto \|g^{-1}h\|;$$

moreover, any such metric is bi-invariant if and only if the related length function is conjugation-invariant. The above observation (1.1) shows that the norm-property of $\|\cdot\|$ automatically implies the bi-invariance of $d(\cdot, \cdot)$.

Using this observation, one can produce another class of *non-abelian* groups which cannot have a norm. Namely, if G is a connected Lie group with a norm – hence a bi-invariant metric – then by Milnor's result [4, Lemma 7.5], $G \cong K \times \mathbb{R}^m$ for some compact Lie group K and integer $m \geqslant 0$. If K is nontrivial then $d(e, k^n) = n \cdot d(e, k)$ grows without bound for any $k \in K \setminus \{e\}$, as $n \to \infty$. So K is compact, yet has infinite diameter, which is impossible. Thus $K = (e)$ and so $G \cong \mathbb{R}^m$ must be abelian.

One can now ask the question: **Must every group with a norm be abelian?**

This question is what motivated the work [5] and hence the present note. As mentioned above, it arose out of previous joint work [3], wherein we showed the Khinchin–Kahane inequality over all normed abelian groups. In the course of this, we showed:

Theorem 1.2. *Every normed abelian group G embeds additively and isometrically into a normed linear space $\mathbb{B}(G)$.*

(In fact, we constructed in [3] the "smallest" Banach space containing G – and also extended Theorem 1.2 to normed abelian semigroups.) Now a natural follow-up – in the spirit of our non-abelian results in [2] – would be to extend the Khinchin–Kahane result, or at least the embedding of G into $\mathbb{B}(G)$, to the setting of non-abelian groups G. Thus, the first step is to seek examples of such groups G. Preliminary investigations into the literature, and private communication with several experts (around 2015) did not reveal any such examples – which led naturally to the above question, which can be reformulated as asking: *do non-abelian normed groups exist at all?*

As mentioned above: no examples of such groups were known, nor was a "complete" negative result. Partial negative results (for finite groups or for connected Lie groups) are mentioned above; it is also not hard to see that non-abelian nilpotent groups cannot be normed. On the other hand, Robert Young explained to us (see [5]) that *every* free monoid can be equipped with a norm, in the spirit of bi-invariant word metrics. Given this, it is a striking contrast that (see the above punchline) there is **no** such non-abelian group:

Theorem 1.3 (D.H.J. Polymath, [5]). *Given a group G, the following are equivalent.*

1. *(Algebra:) G is abelian and torsion-free.*
2. *(Analysis:) G is a metric space with a "norm", i.e. a translation-invariant metric $d_G(\cdot,\cdot)$ satisfying: $d_G(1, g^n) = |n|\, d_G(1, g)$ for all $g \in G$ and integers n.*
3. *(Geometry:) G admits a length function $\ell : G \to [0,\infty)$ such that $\ell(g^2) = 2\,\ell(g)\ \forall g \in G$. (We use $\ell(\cdot)$ instead of $\|\cdot\|$ henceforth.)*
4. *G embeds isometrically and additively in a Banach space.*

Thus, the above examples of additive subgroups of normed linear spaces are the *only* examples of groups with norms. Moreover, in a sense Theorem 1.3 is a slogan for the "unity of mathematics". E.g. it provides a purely analysis-based characterization of a fundamental algebraic object.

The remainder of this note explains the proof of all of the steps in the above equivalence. The nontrivial assertion in here is (3) $\implies$ (1), which will take up most of the proof. For now, we mention that the proof of (3) $\implies$ (1) below can be slightly modified to show a quantitative refinement (for full details, see [5]):

Theorem 1.4 (D.H.J. Polymath, [5]). *Let G be a group, let $c, c' \in \mathbb{R}$, and let $\ell : G \to \mathbb{R}$ be a function satisfying:*

$$\ell(gh) \leqslant \ell(g) + \ell(h) + c, \qquad \ell(g^2) \geqslant 2\ell(g) - c', \qquad \forall g, h \in G.$$

Then one has a uniform upper bound on all commutators $[g,h] := ghg^{-1}h^{-1}$:

$$\ell([g,h]) \leqslant 4c + 5c', \qquad \forall g, h \in G.$$

Setting $\ell(\cdot) = \|\cdot\|$ and $c = c' = 0$ yields (3) $\implies$ (1) in Theorem 1.3.

2. Proofs – first part

Here we show the "easy" implications in Theorem 1.3. Clearly, (4) implies (1)–(3). Next, (2) implies (3); we show the converse. Given the dictionary between left-invariant metrics and length functions, we claim that

$$\ell(g^2) = 2\ell(g) \quad \forall g \in G \tag{2.1}$$

implies $\ell(g^n) = n\ell(g)$ for all $g \in G$ and $n > 0$. Indeed, this follows for n a power of 2 by repeatedly applying (2.1). Now say $n = 6$; the proof – found in [1] – is similar for other $n > 2$. Using sub-additivity and homogeneity,

$$8\ell(g) = \ell(g^8) = \ell(g \cdot g \cdot g^6) \leqslant 2\ell(g) + \ell(g^6) \leqslant 2\ell(g) + 6\ell(g) = 8\ell(g).$$

From this it follows that $\ell(g^6) = 6\ell(g)$; similarly for all $n > 0$. Finally, using (1.1) shows that (3) $\implies$ (2).

We next show that (1) $\implies$ (4); then the outstanding implication would be that (3) (equivalently, (2)) implies (1) – and this will be shown in the next and final section. (Note that if we show (3) $\implies$ (1), then (3) $\implies$ (4) via Theorem 1.2.) Thus, suppose (1) holds: G is abelian and torsion-free. Then G embeds into $G \otimes_{\mathbb{Z}} \mathbb{R}$, which is an $\mathbb{R}$-vector space and hence has a norm $\| \cdot \|$ by Zorn's Lemma – e.g., the L^1-norm with respect to a Hamel basis. Restrict $\| \cdot \|$ to the $\mathbb{Q}$-vector subspace $G \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $\mathbb{B}(G)$ be the Cauchy completion under $\| \cdot \|$ of $G \otimes_{\mathbb{Z}} \mathbb{Q}$, with scalar multiplication and norm defined as follows: given $r \in \mathbb{R}$ and a Cauchy sequence $b = [(x_n)_n] \in \mathbb{B}(G)$, let $q_n \to r$ be a rational sequence, and define

$$r \cdot b := [(q_n x_n)_n], \qquad \|b\| := \lim_{n \to \infty} \|x_n\|.$$

Standard arguments show $\mathbb{B}(G)$ – which contains $(G, \| \cdot \|)$ additively – is a real vector space with a norm, and complete. Thus (1) $\implies$ (4).

3. A Polymath blog, a computer, and the journey of discovery

3.1. **Word-games on Tao's blog.** It remains to show that if G is a normed group – and we will denote the norm by ℓ henceforth – then it is abelian and torsion-free. The latter was explained in the lines preceding (1.1), so it remains to show G is abelian. This was the subject of a blogpost by Tao [6] on Dec 16, 2017 (in UCLA; it was Dec 17 in India), where he invited comments and suggestions on how to deduce that G is abelian.

For completeness: at the time of [6], it was not known if there exist non-abelian normed groups or not, and so a couple of days were spent in trying (unsuccessfully) to come up with examples of such groups. This process turned out to prove that some classes of non-abelian groups did *not* possess a norm: nilpotent groups; and lamplighter and solvable groups, to name two. Then researchers turned to the problem of proving G is abelian if it is normed. The approach that was adopted was as follows: a group is abelian if $[g,h] = ghg^{-1}h^{-1}$ is the identity, if and only if $\ell([g,h]) = 0\ \forall g,h \in G$. Thus, it suffices to fix elements $e \neq g,h \in G$ (a normed group), and show

$$\ell([g,h]) \leqslant \epsilon \quad \forall \epsilon > 0. \tag{3.1}$$

Since g,h are fixed, one may rescale the norm such that $\ell(g^{\pm 1}), \ell(h^{\pm 1}) \leqslant 1$. Now sub-additivity trivially shows (3.1) for $\epsilon = 4$:

$$\ell([g,h]) \leqslant \ell(g) + \ell(h) + \ell(g^{-1}) + \ell(h^{-1}) \leqslant 1+1+1+1 = 4,$$

while conjugation-invariance (1.1) immediately improves this to $\epsilon = 2$:

$$\ell([g,h]) \leqslant \ell(g) + \ell(hg^{-1}h^{-1}) = \ell(g) + \ell(g^{-1}) \leqslant 2.$$

A cleverer argument improves 2 to 4/3. The principle here and below is that to show $\ell([g,h]) \leqslant p/q$ for integers $p,q > 0$ is the same as showing

$$p \geqslant q\ell([g,h]) = \ell([g,h]^q) = \ell(ghg^{-1}h^{-1} \cdots ghg^{-1}h^{-1}).$$

With this in mind, to prove (1.1) for $\epsilon = 4/3$, we use the associativity of the group operation together with conjugation-invariance:

$$\begin{aligned} 3\ell([g,h]) &= \ell(ghg^{-1} \cdot h^{-1}gh \cdot g^{-1}h^{-1}g \cdot hg^{-1}h^{-1}) \\ &\leqslant \ell(ghg^{-1}) + \ell(h^{-1}gh) + \ell(g^{-1}h^{-1}g) + \ell(hg^{-1}h^{-1}) \\ &= \ell(h) + \ell(g) + \ell(h^{-1}) + \ell(g^{-1}) \leqslant 4, \end{aligned} \tag{3.2}$$

and so $\ell([g,h]) \leqslant 4/3$ as desired.

These computations were already in the blogpost [6], and subsequent comments there improved the upper bound ϵ on the commutator length $\ell([g,h])$, from $\epsilon = 4/3$ (Dec 17 in India) to $\epsilon = 5/4, 19/16$, and eventually to 22/23 – at 11:12 am on Dec 20. (Here and below, we record some of the timepoints – all in Indian Standard Time – to give an idea of the speed with which the research progressed.) Each of these improvements was achieved using successively more involved and complicated "word games" of the form (3.2).

Then, at this point there was almost a 24-hour break in progress, perhaps owing to the increasingly complex computations needed to improve the upper bound ϵ. It was not clear how to proceed at this point.

3.2. **Computer-assisted proof.** It was at this point that the proof went from being "crowdsourced" to also being "computer-assisted". S. Gadgil (at IISc Bangalore, and one of the authors of the paper) had programmed his computer to search through a large number of words, using what it learned along the way to improve the speed of the search. The computer was able to reduce the bound ϵ from $22/23 \approx 0.957$ to 0816. Gadgil posted the proof of this bound (which he had programmed the computer to write out – it took 120+ steps!), and this proof was then taken apart, analyzed, and the key underlying idea isolated into the following result:

Lemma 3.1 (Internal repetition trick, [5])**.** *Suppose G is a normed group and $x, y, z, w \in G$ are such that x is conjugate to wy and to zw^{-1}. Then*

$$2\ell(x) \leqslant \ell(y) + \ell(z) \qquad (\textit{this does not depend on } w).$$

Proof. Suppose $x = s(wy)s^{-1} = t(zw^{-1})t^{-1}$. We compute:

$$\begin{aligned}\ell(x^n \cdot x^n) &= \ell(s(wy)^n s^{-1} \cdot t(zw^{-1})^n t^{-1}) \\ &\leqslant \ell(s) + \ell(t^{-1}) + \ell((wy)^n s^{-1} t(zw^{-1})^n).\end{aligned}$$

Now note that the final (long) word on the right is a conjugate, of the form $w\mathbf{t}_0 w^{-1}$ for some word $\mathbf{t}_0$. Hence its norm equals $\ell(\mathbf{t}_0)$. But $\mathbf{t}_0 = y \cdot w\mathbf{t}_1 w^{-1} \cdot z$ for some $\mathbf{t}_1$, so we play the same game using sub-additivity:

$$\ell(x^{2n}) \leqslant \ell(s) + \ell(t^{-1}) + \ell(y) + \ell(z) + \ell(w\mathbf{t}_1 w^{-1}).$$

Repeatedly expanding as above, we obtain:

$$2n\,\ell(x) = \ell(x^{2n}) \leqslant n(\ell(y) + \ell(z)) + (\ell(s) + \ell(s^{-1}) + \ell(t) + \ell(t^{-1})).$$

Now divide both sides by n and let $n \to \infty$. □

Before proceeding further, we show a sample application of (the power of) Lemma 3.1. For completeness, here is the timeline: after the bound of 22/23 was posted, nearly a day later Gadgil posted the improved bound – using his computer – of 0.816 on Dec 21 at 9:30 am. The proof (by Gadgil's computer) was taken apart and slightly improved by 11:10 am. A few hours later by 1:45 pm, the above Lemma 3.1 had been formulated, proved, and also *applied* to obtain an improved bound of 8/11 – all on Tao's blog:

Corollary 3.2. *Let (G,ℓ) be a normed group, and fix nontrivial elements $\alpha,\beta\in G$; rescale such that $\ell(\alpha^{\pm 1}),\ell(\beta^{\pm 1})\leqslant 1$. Then $\ell([\alpha,\beta])\leqslant 8/11$.*

Proof. Specialize Lemma 3.1 to

$$x=[\alpha,\beta]^2\alpha=\alpha\beta\overline{\alpha}\overline{\beta}\alpha\beta\overline{\alpha}\overline{\beta}\alpha,$$
$$y=\overline{\alpha}(\overline{\beta}\alpha\beta\cdot\overline{\alpha}\overline{\beta}\alpha)\alpha,\quad w=\beta,\quad z=\alpha(\alpha\beta\overline{\alpha}\cdot\overline{\beta}\alpha\beta)\overline{\alpha},$$

where $\overline{\alpha}=\alpha^{-1},\overline{\beta}=\beta^{-1}$. Thus x is conjugate to wy and to zw^{-1}, and so

$$\ell(x)=\ell([\alpha,\beta]^2\alpha)\leqslant\frac{1}{2}(\ell(y)+\ell(z))\leqslant\frac{2+2}{2}=2$$

by the lemma. Similarly, $\ell([\beta,\overline{\alpha}]^2\beta)\leqslant 2$, etc. But this shows the result:

$$\begin{aligned}11\,\ell([\alpha,\beta])&=\ell[\alpha(\beta\overline{\alpha}\overline{\beta}\alpha\beta\overline{\alpha}\overline{\beta}\alpha\beta)\overline{\alpha}\cdot\overline{\beta}(\alpha\beta\overline{\alpha}\overline{\beta}\alpha\beta\overline{\alpha}\overline{\beta}\alpha)\beta\\&\qquad\cdot\overline{\alpha}(\overline{\beta}\alpha\beta\overline{\alpha}\overline{\beta}\alpha\beta\overline{\alpha}\overline{\beta})\alpha\cdot\beta(\overline{\alpha}\overline{\beta}\alpha\beta\overline{\alpha}\overline{\beta}\alpha\beta\overline{\alpha})\overline{\beta}\ ]\\&\leqslant\ell([\beta,\overline{\alpha}]^2\beta)+\ell([\alpha,\beta]^2\alpha)+\ell([\overline{\beta},\alpha]^2\overline{\beta})+\ell([\overline{\alpha},\overline{\beta}]^2\overline{\alpha})\\&\leqslant 2+2+2+2=8.\end{aligned}$$

□

Nevertheless, notice that each improvement from one finite number to the next, is also a *logarithmically* finite improvement – whereas we want to take $\log(\epsilon)$ all the way down to $\log(0)=-\infty$. Thus, sample applications like the above computation for 8/11, will not suffice to prove the theorem.

3.3. **Probability to the rescue.** The next – and final – input at this point that is needed to complete the proof of Theorem 1.3 (i.e., that every normed group G is abelian – i.e. in turn that $\ell([\alpha,\beta])=0\ \forall\alpha,\beta\in G$) is binomial combinatorics, which is compactly expressed using the language of probability theory. We begin with a different application of Lemma 3.1, which emerged a few hours later, on 22 Dec at 12:27 am on Tao's blog.

Corollary 3.3. *Let (G,ℓ) be a normed group. Fix $\alpha,\beta\in G$ and define*

$$f=f_{\alpha,\beta}:\mathbb{Z}^2\to\mathbb{R},\qquad f(m,k):=\ell(\alpha^m[\alpha,\beta]^k).$$

Then for all $m,k\in\mathbb{Z}$, we have:

$$f(m,k)\leqslant\frac{1}{2}(f(m-1,k)+f(m+1,k-1)).\tag{3.3}$$

Proof. Note that $\alpha^m[\alpha,\beta]^k$ is conjugate to both $\alpha(\alpha^{m-1}[\alpha,\beta]^k)$ and to $(\beta^{-1}\alpha^m[\alpha,\beta]^{k-1}\alpha\beta)\alpha^{-1}$. Hence by Lemma 3.1,

$$\ell(\alpha^m[\alpha,\beta]^k)\leqslant\frac{\ell(\alpha^{m-1}[\alpha,\beta]^k)+\ell(\beta^{-1}\alpha^m[\alpha,\beta]^{k-1}\alpha\beta)}{2}.$$

Since $\beta^{-1}\alpha^m[\alpha,\beta]^{k-1}\alpha\beta$ is conjugate to $\alpha^{m+1}[\alpha,\beta]^{k-1}$, Lemma 3.1 now gives (3.3). $\square$

The trick now is to apply (3.3) $2n$ times to bound $\ell(([\alpha,\beta])^n)$ – for large n. We write out the $n=2$ special case here, to illustrate the workings:

$$\begin{aligned}\ell([\alpha,\beta]^2) = f(0,2) &\leqslant \frac{f(-1,2)+f(1,1)}{2} \leqslant \frac{f(-2,2)+2f(0,1)+f(2,0)}{4}\\ &\leqslant \frac{1}{8}(f(-3,2)+3f(-1,1)+3f(1,0)+f(3,-1))\\ &\leqslant \frac{1}{16}(f(-4,2)+4f(-2,1)+6f(0,0)+4f(2,-1)+f(4,-2)).\end{aligned}$$

Similarly, for any $n \geqslant 1$ one can show that

$$\ell([\alpha,\beta]^n) \leqslant \frac{1}{2^{2n}}\sum_{j=-n}^{n}\binom{2n}{j+n} f\left(2j(1,-\frac{1}{2})\right). \tag{3.4}$$

We write this using probabilistic notation. Let $Y=\pm 1$ with probability $1/2$ each – i.e., a *Rademacher* variable. Then (3.3) can be rewritten as:

$$f(m,k) \leqslant \mathbb{E} f\left((m,k-\frac{1}{2}) + Y(1,-\frac{1}{2})\right),$$

and the key is the drift of $(0,-1/2)$. Similarly, applying (3.3) twice yields

$$f(m,k) \leqslant \mathbb{E} f\left((m,k-1) + (Y_1+Y_2)(1,-\frac{1}{2})\right),$$

where Y_1, Y_2 are i.i.d. Rademacher variables. Repeat this $2n$ times, starting with $f(0,n)=\ell([\alpha,\beta]^n)$, to obtain (3.4) in probabilistic form:

$$f(0,n) = \ell([\alpha,\beta]^n) \leqslant \mathbb{E} f\left((Y_1+\cdots+Y_{2n})(1,-\frac{1}{2})\right),$$

with $Y_1,\dots,Y_{2n}$ i.i.d. Rademacher variables. (Thus $Y_1+\cdots+Y_{2n}$ is even, so the above is well-defined.) Set $\mathcal{Y} := Y_1+\cdots+Y_{2n}$. By sub-additivity,

$$0 \leqslant f(\mathcal{Y}(1,-1/2)) = \ell(\alpha^{\mathcal{Y}}[\alpha,\beta]^{-\mathcal{Y}/2}) \leqslant A|\mathcal{Y}|,$$

for some $A>0$ (specifically, $A=\ell(\alpha)+\frac{1}{2}\ell([\alpha,\beta]^{-1})$). Hence,

$$f(0,n) \leqslant \mathbb{E} f(\mathcal{Y}(1,-1/2)) \leqslant A\cdot\mathbb{E}[|\mathcal{Y}|] \leqslant A\cdot\mathbb{E}[\mathcal{Y}^2]^{1/2},$$

the last by Jensen's inequality. But $\mathcal{Y}$ has mean zero and – the key point! – has variance $2n$ (which grows slower than quadratically in n). Therefore

$$\ell([\alpha,\beta]^n) = f(0,n) \leqslant A\cdot\mathbb{E}[\mathcal{Y}^2]^{1/2} = A\cdot\sqrt{2n}.$$

Finally, divide by n and let $n \to \infty$; thus $\ell([\alpha, \beta]) = 0$. Hence $[\alpha, \beta] = \alpha\beta\overline{\alpha}\overline{\beta} = e$ for all $\alpha, \beta \in G$, which implies G is abelian, as claimed. $\square$

Remark 3.4. The preceding argument using the linear growth of the variance was not the original one used; in fact Tao showed this last step using a Chernoff bound, on his blog at 3:57 am on 22 Dec. It was when the paper was being written up later, that the above, simpler argument was found.

Theorem 1.3 has a quantitative refinement (see Theorem 1.4) as well as some applications to geometric group theory involving the stable commutator length, for which the reader is referred to [5]. Eventually, [5] was published under the pseudonym of "Density Hales Jewett Polymath", or "D.H.J. Polymath" in short. In addition to the elegant and clean characterization of abelian torsion-free groups via analysis, the journey of discovery was a "modern" one – completed in just five days! – and involving crowdsourcing, a blog, and a computer to aid human efforts. Hopefully, this as-yet-nonstandard model of mathematical research will continue to grow and to flourish, and will aid mathematical discovery in the years ahead.

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GC^1-POSITIVITY AND MONOTONICITY PRESERVING INTERPOLATION USING RATIONAL CUBIC TRIGONOMETRIC SPLINE

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Abstract. In this paper, we study a GC^1 rational cubic trigonometric spline and develop positivity, monotonicity and constrained curve interpolation schemes by using a GC^1 piecewise rational cubic trigonometric spline with five shape parameters. The approximation properties are discussed and confirmed that the expected approximation order is h^2.

1. Introduction

At present, spline functions have become the main tool for solving the majority of problems involving the approximation of functions, which also includes interpolation problems. Shape-preserving interpolation is a powerful tool to visualize the data in the form of curves and surfaces. The shape characteristics like positivity, monotonicity, and convexity can be easily observed when data arises from physical experiments. In this case, it becomes vital that the interpolant produces curves smoother and represents physical reality as close as possible. For this purpose, designers and engineers want approximation methods that represent physical situations accurately. In recent years, polynomial splines and NURBS are replaced by trigonometric splines to prevail over the difficulties faced using the former. Several authors have studied trigonometric splines to represent curves and surfaces (see [1], [2], [6], [7], [11], [12]). The significance of the trigonometric splines in other areas, such as electronics and medicine are acknowledged in Hoschek and Lasser [4]. The trigonometric splines have gained some interest within CAGD especially in CD. Schoenberg [11] has introduced

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the trigonometric B-spline. The cubic trigonometric Bezier curve with two shape parameters was introduced by Xi-An Han et al. [3]. Karim [5] has proposed GC^1 monotonicity preserving using ball cubic interpolation. The above discussed shape preserving properties of trigonometric splines motivated us to construct GC^1 monotonicity preserving interpolation using cubic trigonometric spline.

In this paper, we present a GC^1 piecewise rational cubic trigonometric interpolating curve scheme for positive, monotone and constrained data. The approximation properties of GC^1 rational cubic trigonometric spline have discussed.

2. GC^1 Piecewise Rational Cubic Trigonometric Spline

In this section, a GC^1 piecewise rational cubic trigonometric spline with shape parameters developed. Let $[a, b]$ be the given interval such that $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ and $h_i = t_{i+1} - t_i$, $i = 0, 1, 2, \dots n-1$. f_i's are the given data at knots t_i for all i. The GC^1 piecewise rational cubic trigonometric function with four parameters over each subinterval $I_i = [t_i, t_{i+1}]$, $i = 0, 1, 2, \dots n-1$ is defined as:

$$P(t) = P_i(t) = \frac{p_i(t)}{q_i(t)} \tag{2.1}$$

$$\begin{aligned}
p_i(t) &= \alpha_i f_i (1-\sin\theta)^3 \\
&\quad +2\left(\frac{\alpha_i h_i d_i}{\pi r_i} + \beta_i f_i\right)\sin\theta(1-\sin\theta)^2 \\
&\quad +2\left(\gamma_i f_{i+1} - \frac{\delta_i h_i d_{i+1}}{\pi r_i}\right)\cos\theta(1-\cos\theta)^2 \\
&\quad +\delta_i f_{i+1}(1-\cos\theta)^3 \\
q_i(t) &= \alpha_i(1-\sin\theta)^3 + 2\beta_i\sin\theta(1-\sin\theta)^2 \\
&\quad +2\gamma_i\cos\theta(1-\cos\theta)^2 + \delta_i(1-\cos\theta)^3
\end{aligned}$$

Where $\theta = \frac{\pi}{2}(\frac{t-t_i}{h_i})$, $\theta \in [0, \frac{\pi}{2}]$ and α_i, β_i, γ_i, δ_i and r_i are positive free shape parameters and the spline in (2.1) is GC^1-continuous which satisfies the following interpolating conditions:

$$P(t_i) = f_i,\ P(t_{i+1}) = f_{i+1},\ P'(t_i) = \frac{d_i}{r_i},\ P'(t_{i+1}) = \frac{d_{i+1}}{r_i} \tag{2.2}$$

$P'(t)$ denotes the derivative with respect to t and d_i are derivative values at given knots t_i. These d_i either given or can be computed by some numerical method [8]. The parameters r_i are positive, when $r_i = 1$ the rational cubic trigonometric spline interpolant (2.1) reduces to C^1 rational cubic trigonometric spline. Equation (2.1) can be written in the form

$$P_i(t) = R_0 f_i + R_1 U_i + R_2 V_i + R_3 f_{i+1} \tag{2.3}$$

where

$$\begin{aligned} R_0 &= \frac{\alpha_i(1-\sin\theta)^3}{R} \\ R_1 &= \frac{2\beta_i \sin\theta(1-\sin\theta)^2}{R} \\ R_2 &= \frac{2\gamma_i \cos\theta(1-\cos\theta)^2}{R} \\ R_3 &= \frac{\delta_i(1-\cos\theta)^3}{R} \end{aligned}$$

$$\begin{aligned} R = {} & \alpha_i(1-\sin\theta)^3 + 2\beta_i \sin\theta(1-\sin\theta)^2 \\ & + 2\gamma_i \cos\theta(1-\cos\theta)^2 + \delta_i(1-\cos\theta)^3 \end{aligned}$$

and

$$\begin{aligned} U_i &= \left(\frac{\alpha_i h_i d_i}{\pi r_i} + \beta_i f_i\right) \\ V_i &= \left(\gamma_i f_{i+1} - \frac{\delta_i h_i d_{i+1}}{\pi r_i}\right) \end{aligned}$$

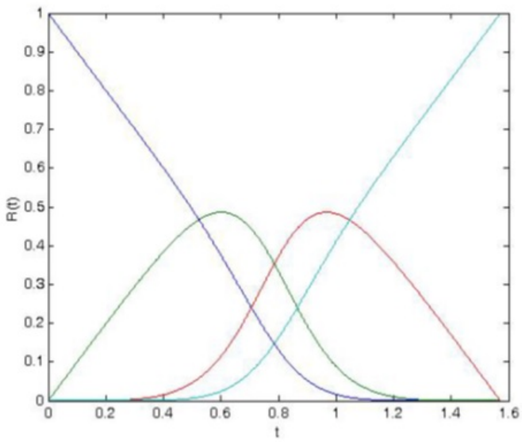

FIGURE 1. The graph of $R_i(t)$

R_i's (See Figure 1) are appropriately defined rational functions with following properties:

(i) Nonnegativity: $R_i(t) \geq 0$, for $t \in [t_i,\ t_{i+1}]$ or $\theta \in [0,\ \frac{\pi}{2}]$

We observe that

$$\sin\theta \geq 0, \qquad (1-\sin\theta) \geq 0,$$
$$\cos\theta \geq 0, \qquad (1-\cos\theta) \geq 0$$

$\Rightarrow \quad R_i(t) \geq 0, \quad \text{for } i = 0,\ 1,\ 2,\ 3.$

(ii) Partition of unity: $\sum_{i=0}^{3} R_i = 1$

(iii) Monotonicity: For given parameter t, R_0, R_3 are monotonically increasing and R_1, R_2 are monotonically decreasing in interval $\left(\frac{\pi}{9},\ \frac{\pi}{2}\right)$ and $\left(\frac{4\pi}{9},\ \frac{\pi}{2}\right)$ respectively (see Figure 1) in R^N, $N > 1$, and for α_i, β_i, γ_i and $\delta_i > 0$.

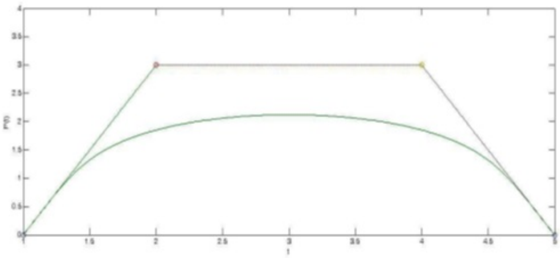

FIGURE 2. The graph of $P_i(t)$

The rational cubic trigonometric function defined by (2.1) has following properties:

(i) Convex hull property: The entire curve segment lies inside its control polygon spanned by P_0, P_1, P_2 and P_3. See Figure 2.

(ii) Variation diminishing property: The curve segment P_i crosses any plane of dimension $N-1$ no more times than it crosses the control polygon joining f_i, V_i, W_i and f_{i+1}.

3. Positive Curve Interpolation

Positivity is very significance aspect of the shape in the computer graphics and computer aided geometric design. In this section, we utilize GC^1 piecewise rational cubic trigonometric spline which is developed in section 2 to generate a positivity preserving curve using a positive data set.

Consider a data set $\{(t_i,\ f_i)\ :\ i = 0,\ 1, \ldots, n\}$ such that

$$f_i > 0, \quad i = 0,\ 1,\ 2, \ldots, n \tag{3.1}$$

GC^1 piecewise rational cubic trigonometric spline given in (2.1) preserves positivity through positive data if $P_i(t) > 0,\ t \in [t_i,\ t_{i+1}],\ i = 0,\ 1,\ \ldots,\ n-1$. We observe that the denominator $q_i(t)$ in (2.1) is positive because the shape parameters $\alpha_i,\ \beta_i,\ \gamma_i$ and δ_i are all positive. It remains to show that the numerator $p_i(t)$ in (2.1) is positive. The numerator $p_i(t) > 0$ if all its coefficients are positive. It yields

$$\left.\begin{array}{rcl} \beta_i & > & \frac{-d_i h_i \alpha_i}{\pi r_i f_i} \\ \gamma_i & > & \frac{d_{i+1} h_i \delta_i}{\pi r_i f_{i+1}} \end{array}\right\} \tag{3.2}$$

The sufficient conditions for the interpolant defined in (2.1) are to preserve the positivity of positive data if shape parameters satisfy

$$\left.\begin{array}{rcl} \alpha_i, \delta_i & > & 0 \\ \beta_i & > & \max\{0,\ \frac{-d_i h_i \alpha_i}{\pi r_i f_i}\} \\ \gamma_i & > & \max\{0,\ \frac{d_{i+1} h_i \delta_i}{\pi r_i f_{i+1}}\} \end{array}\right\} \tag{3.3}$$

Thus we have proved the following theorem.

Theorem 3.1. *The sufficient conditions for the interpolant in* (2.1) *to preserve the positivity of positive data are that the shape parameters are satisfy the following conditions:*

$$\begin{array}{rcl} \alpha_i, \delta_i & > & 0 \\ \beta_i & > & \max\{0,\ \dfrac{-d_i h_i \alpha_i}{\pi r_i f_i}\} \\ \gamma_i & > & \max\{0,\ \dfrac{d_{i+1} h_i \delta_i}{\pi r_i f_{i+1}}\} \end{array}$$

In order to examine the above theorem, let us take the 2D positive data set as in Table 1

i	1	2	3	4	5	6	7	8
t_i	0	.04	.05	.06	.07	.08	.12	.13
f_i	.82	1.2	.978	.6	.3	.1	.15	.48

TABLE 1. A 2D positive data set

The developed scheme is used to demonstrate the positivity preserving of positive data. Random values to the shape parameters are assigned and it is clearly visible that the resulting curve do not preserve the positivity see figure 3. On the other hand, the positivity preserving curve in figure 4 is generated by the theorem 3.1.

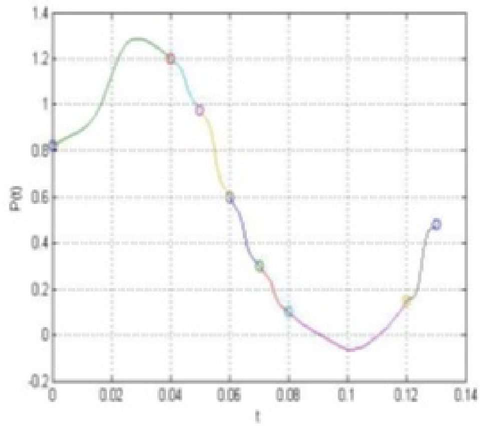

FIGURE 3. Non-positivity preserving rational cubic trigonometric curve with $\alpha_i = 0.7,\ \beta_i = 0.2,\ \gamma_i = 0.2,\ \delta_i = 0.7$ and $r_i = 1$

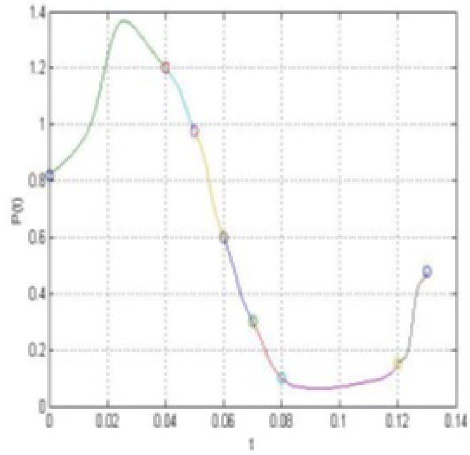

FIGURE 4. A GC^1 Positivity preserving rational cubic trigonometric curve with $\alpha_i = 0.5,\ \delta_i = 0.5$ and $r_i = 1.2$

4. Monotonicity Preserving Interpolation

This section discusses a monotonicity preserving curve interpolating scheme with four parameters with given set of monotone data points.

Let $(t_i,\ f_i),\ i = 0,\ 1,\ \ldots,\ n$ be a given monotone data. Thus for monotone increasing data (use same approach for monotone decreasing data) with

$$f_i \ \leq \ f_{i+1},\ i = 0,\ 1,\ 2,\ \ldots,\ n-1 \tag{4.1}$$

Theorem 4.1. *The GC^1 piecewise rational cubic trigonometric spline defined in* (2.1) *preserves the monotonicity through monotone data in each subinterval* $[t_i, t_{i+1}]$, $i = 0,\ 1,\ 2,\ \ldots,\ n-1$. *if the shape parameters satisfy the following conditions;*

$$\left.\begin{array}{l} \alpha_i,\ \delta_i\ > 0 \\ u_i < \beta_i < v_i \\ w_i < \gamma_i < x_i \end{array}\right\} \tag{4.2}$$

where

$$\left.\begin{array}{ll} u_i = \frac{d_i \alpha_i}{\pi r_i \Delta_i}, & v_i = \max\{\frac{3\alpha_i}{2},\ (\frac{3}{2} + \ \frac{d_i}{\pi r_i \Delta_i})\alpha_i\} \\ w_i = \frac{d_{i+1}\delta_i}{\pi r_i \Delta_i}, & x_i = \max\{\frac{3\delta_i}{2},\ (\frac{3}{2}\ +\ \frac{d_{i+1}}{\pi r_i \Delta_i})\delta_i\} \end{array}\right\} \tag{4.3}$$

Proof. Let $\{(t_i,\ f_i):\ i = 0,\ 1,\ 2, \ldots,\ n\}$ be monotonically increasing data set. That is $f_i \leq f_{i+1}$ or equivalently

$$\Delta_i \ = \ \frac{f_{i+1}\ -\ f_i}{h_i}\ \geq\ 0 \tag{4.4}$$

Also, the necessary condition for monotonicity is that

$$d_i\ \geq\ 0,\ i = 0,\ 1,\ 2,\ \ldots,\ n \tag{4.5}$$

It arises the following two cases for the interpolant (2.1) to preserve the monotonicity of monotone data.

Case 1: $\mathrm{d}_i = \mathrm{d}_{i+1} = 0$ when $\Delta_i = 0$. In this case, $\mathrm{P}_i(\mathrm{t})$ reduces to

$$P_i(t)\ =\ f_i,\ \forall\ [t_i,\ t_{i+1}] \tag{4.6}$$

This proves that the interpolant is monotonic.

Case 2: When $\Delta_i\ \neq\ 0$, then $P_i(t)$ is monotonically increasing if and only if

$$P_i^{'}(t)\ >\ 0,\ \forall\ [t_i,\ t_{i+1}] \tag{4.7}$$

$P_i^{'}(t)$ is presented in simpler form as

$$\begin{aligned} P_i^{'}(t) \quad = \quad & \frac{\pi}{2h_i[q_i(t)]^2} \times A_1 \cos\theta \sin\theta (1-\sin\theta)^4 \\ & + A_2 (1-\cos\theta)^2 \sin\theta (1-\sin\theta)^3 \\ & + A_3 \cos\theta (1-\cos\theta) \sin\theta (1-\sin\theta)^3 \end{aligned}$$

$$\begin{aligned}
&+A_4(1-\cos\theta)^2\sin^2\theta(1-\sin\theta)^2\\
&+A_5(1-\sin\theta)^2\cos^2\theta(1-\cos\theta)^2\\
&+A_6\sin\theta(1-\sin\theta)\cos\theta(1-\cos\theta)^3\\
&+A_7(1-\sin\theta)^2\cos\theta(1-\cos\theta)^3\\
&+A_8\sin\theta\cos\theta(1-\cos\theta)^4\\
&+A_9\cos^2\theta(1-\cos\theta)^2\sin\theta(1-\sin\theta)\\
&+A_9\sin^2\theta(1-\sin\theta)^2\cos\theta(1-\cos\theta) \qquad (4.8)
\end{aligned}$$

with

$$\begin{aligned}
A_1 &= \tfrac{2d_i\alpha_i^2}{\pi r_i}, \quad A_2 = \left(\left(-2\gamma_i+3\delta_i\right)\Delta_i + \tfrac{2d_{i+1}\delta_i}{\pi r_i}\right)\alpha_i\\
A_3 &= \left(\gamma_i\Delta_i - \tfrac{d_{i+1}\delta_i}{\pi r_i}\right)4\alpha_i\\
A_4 &= 2\left(\beta_i\Delta_i\left(3\delta_i-2\gamma_i\right) + \tfrac{2\beta_i\delta_i d_{i+1}}{\pi r_i} + \tfrac{(-3\delta_i+2\gamma_i)d_i\alpha_i}{\pi r_i}\right)\\
A_5 &= 2\left(\gamma_i\Delta_i\left(3\alpha_i-2\beta_i\right) + \tfrac{2\gamma_i\alpha_i d_i}{\pi r_i} + \tfrac{(3\alpha_i-2\beta_i)\delta_i d_{i+1}}{\pi r_i}\right)\\
A_6 &= \left(\beta_i\Delta_i - \tfrac{d_i\alpha_i}{\pi r_i}\right)4\delta_i\\
A_7 &= \left(\left(-2\beta_i+3\alpha_i\right)\Delta_i + \tfrac{2d_i\alpha_i}{\pi r_i}\right)\delta_i\\
A_8 &= \tfrac{2d_{i+1}\delta_i^2}{\pi r_i}, \quad A_9 = 8\left(\beta_i\gamma_i\Delta_i - \tfrac{\alpha_i\gamma_i d_i}{\pi r_i} - \tfrac{\beta_i\Delta_i d_{i+1}}{\pi r_i}\right)
\end{aligned}$$

The denominator of (4.8) is always positive. Thus the sufficient conditions for monotonicity preserving curve are

$$A_i \geq 0, \quad i = 1,\ 2,\ \ldots,\ 9$$

Because $A_1,\ A_8 > 0$.
Also $A_i \geq 0, \quad i = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 9$ if

$$\begin{aligned}
\gamma_i &< \left(\frac{3}{2}+\frac{d_{i+1}}{\pi r_i\Delta_i}\right)\delta_i, \quad \gamma_i < \frac{3\delta_i}{2}\\
\beta_i &< \frac{3\alpha_i}{2}, \quad \frac{d_i\alpha_i}{\pi r_i\Delta_i} < \beta_i, \quad \frac{d_{i+1}\delta_i}{\pi r_i\Delta_i} < \gamma_i,\\
\beta_i &< (\frac{3}{2}+\frac{d_i}{\pi r_i\Delta_i})\alpha_i, \quad \frac{2d_{i+1}\delta_i}{\pi r_i\Delta_i} < \gamma_i \qquad (4.9)
\end{aligned}$$

Hence, to preserve the monotonicity of monotone data and control the shape of the curve as per desire (4.9) can be written as

$$\begin{aligned} \alpha_i,\ \delta_i &> 0 \\ u_i < \beta_i &< v_i \\ w_i < \gamma_i &< x_i \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} u_i &= \tfrac{d_i\alpha_i}{\pi r_i\Delta_i}, \quad v_i = \max\{\tfrac{3\alpha_i}{2},\ (\tfrac{3}{2} + \tfrac{d_i}{\pi r_i\Delta_i})\alpha_i\} \\ w_i &= \tfrac{d_{i+1}\delta_i}{\pi r_i\Delta_i}, \quad x_i = \max\{\tfrac{3\delta_i}{2},\ (\tfrac{3}{2} + \tfrac{d_{i+1}}{\pi r_i\Delta_i})\delta_i\} \end{aligned} \tag{4.11}$$

as required.

□

5. Constrained Curve Interpolation

In this section, it is assumed that the above data is under consideration for any arbitrary line $y = mx + c$.

Theorem 5.1. *The GC^1 piecewise rational cubic trigonometric spline defined in* (2.1) *preserves the shape of data lying above an arbitrary straight line in each subinterval* $[t_i,\ t_{i+1}],\ i = 0,\ 1,\ 2,\ \ldots,\ n-1$, *if the following conditions are satisfied:*

$$\left.\begin{aligned} \alpha_i,\ \delta_i &> 0 \\ \beta_i &> \max\{0,\ -\tfrac{d_i h_i\alpha_i}{\pi r_i(f_i - l)}\} \\ \gamma_i &> \max\{0,\ -\tfrac{d_{i+1} h_i\delta_i}{\pi r_i(f_{i+1} - l)}\} \end{aligned}\right\} \tag{5.1}$$

Proof. Let $\{(t_i,\ f_i) :\ i = 0,\ 1,\ 2,\ \ldots,\ n\}$ be a set of data points lying above a given straight line $y = mx + c$ ie

$$f_i \ > \ mt_i + c \tag{5.2}$$

The curve will be above the straight line if the rational cubic trigonometric spline (2.1) satisfies the following condition

$$P(t) \ > \ mt + c, \quad \forall\ t \in [t_0,\ t_n] \tag{5.3}$$

For each subinterval $[t_i,\ t_{i+1}]$, (5.3) can be expressed as

$$P_i(t) \ = \ \frac{p_i(t)}{q_i(t)} \ > \ \lambda_i\left(1 - \frac{2\theta}{\pi}\right) - \frac{2\theta\eta_i}{\pi} \tag{5.4}$$

$$p_i(t) - lq_i(t) \ > \ 0 \tag{5.5}$$

$$l = \lambda_i\left(1 - \frac{2\theta}{\pi}\right) - \frac{2\theta\eta_i}{\pi}$$

with

$$\lambda_i = mt_i + c$$

and

$$\eta_i = mt_{i+1} + c$$

The equation (5.5) can be written in a simplified form as follows:

$$B_1(1-\sin\theta)^3 + B_2 2\sin\theta(1-\sin\theta)^2 + B_3 2\cos\theta(1-\cos\theta)^2 + B_4(1-\cos\theta)^3 > 0 \qquad (5.6)$$

where

$$B_1 = (f_i - l)\alpha_i$$

$$B_2 = \frac{\alpha_i h_i d_i}{\pi r_i} + \beta_i f_i - l\beta_i$$

$$B_3 = \gamma_i f_{i+1} - \frac{\delta_i h_i d_{i+1}}{\pi r_i} - l\gamma_i$$

$$B_4 = (f_{i+1} - l)\delta_i$$

The equation (5.6) is true if $B_j > 0$ for $j = 1, 2, 3, 4$
As $B_1, B_4 > 0$, $B_2 > 0$, if

$$\beta_i > \frac{-\alpha_i h_i d_i}{\pi r_i(f_i - l)}$$

Also $B_2 > 0$, if

$$\gamma_i > \frac{-d_{i+1} h_i \delta_i}{\pi r_i(f_{i+1} - l)}$$

Thus for a curve constrained by a parameters must satisfy

$$\alpha_i,\ \delta_i > 0, \beta_i > \max\{0,\ -\frac{d_i h_i \alpha_i}{\pi r_i(f_i - l)}\}, \gamma_i > \max\{0,\ -\frac{d_{i+1} h_i \delta_i}{\pi r_i(f_{i+1} - l)}\}$$

as required. □

The data set in Table 2 lies above the line $y = 0.06x + 0.02$. Figure 5 are produced by taking the values of the shape parameters on trial and

I	1	2	3	4	5
t_i	0	1.1	2	3	4.5
f_i	1.5	.4	4	6.2	6

Table 2. 2D data set lying above the line $y = 0.06x + 0.02$

error basis. This figure depict that the curve do not lie above the respective given straight line. On the other hand Figure 6 is generated by using the constrained curve scheme developed in the theorem 5.1.

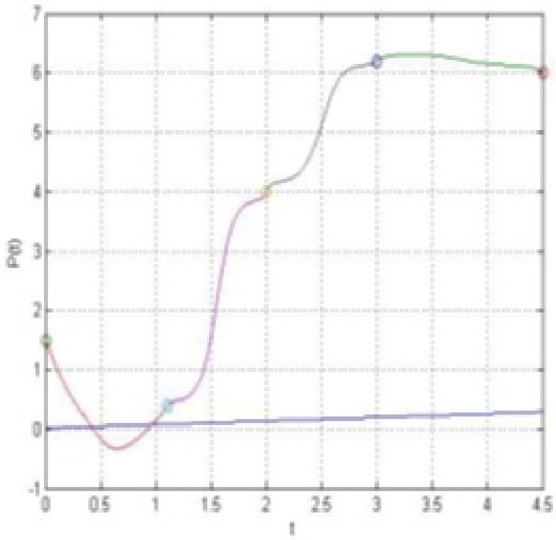

FIGURE 5. A GC^1 rational cubic trigonometric curve which is not totally above the given line with $\alpha_i = .5,\ \beta_i = .1,\ \gamma_i = .1,\ \delta_i = .5$ and $r_i = .6$ without using the theorem 5.1

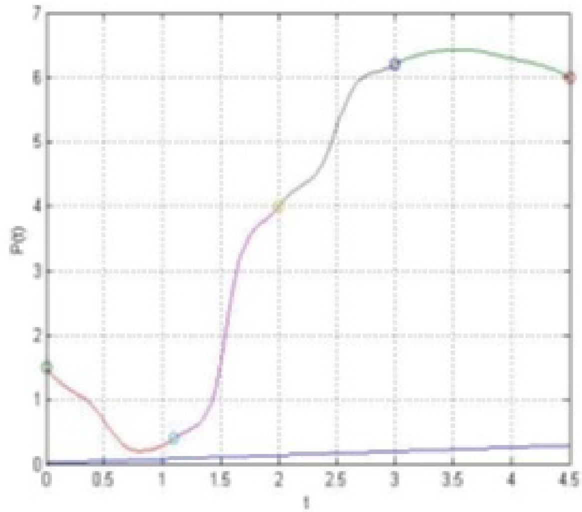

FIGURE 6. A GC^1 rational cubic trigonometric curve lying above the given line with $\alpha = .5,\ \beta_i = .7,\ \gamma_i = .7,\ \delta_i = .5$ and $r_i = 1.2$ after using the theorem 5.1

6. Approximation Property of $GC1$ Cubic Trigonometric Interpolation

To estimate error of the GC^1 cubic trigonometric interpolation function defined by (2.1) since the interpolation is local without loss of generality, we consider the error in the sub interval $[t_i,\ t_{i+1}]$. Consider the case that the knots are equally spaced, namely $h_i = h = \frac{(t_n - t_0)}{n}$ for all $i = 1,\ 2,\ \ldots,\ n$ the Peano Kernel Theorem [10] is used to estimate the error in each subinterval $[t_i,\ t_{i+1}]$ as:

$$R[f] = f(t) - P(t) = \int_{t_i}^{t_{i+1}} f^2(\tau) R_t[(t-\tau)_+] d\tau,\ t \in [t_i,\ t_{i+1}] \qquad (6.1)$$

where

$$R_t[(t-\tau)_+] = \begin{cases} p(\tau), & t_i < \tau < t; \\ q(\tau), & t < \tau < t_{i+1}. \end{cases}$$

where

$$p(\tau) = (t-\tau) - \left\{ \begin{array}{l} (t_{i+1} - \tau)\,[\delta_i (1-\cos\theta)^3 \\ +2\cos\theta (1-\cos\theta)^2] \\ -2\cos\theta(1-\cos\theta)^2 \frac{h_i\delta_i}{\pi r_i} \end{array} \right\} / R \qquad (6.2)$$

$$q(\tau) = \left\{ \begin{array}{l} -(t_{i+1} - \tau)\,[\delta_i (1-\cos\theta)^3 \\ +2\cos\theta (1-\cos\theta)^2] \\ -2\cos\theta(1-\cos\theta)^2 \frac{h_i\delta_i}{\pi r_i} \end{array} \right\} / R \qquad (6.3)$$

$$\begin{aligned} R &= \alpha_i(1-\sin\theta)^3 + 2\beta_i \sin\theta(1-\sin\theta)^2 \\ &\quad +2\gamma_i\cos\theta(1-\cos\theta)^2 + \delta_i(1-\cos\theta)^3 \end{aligned}$$

$$\begin{aligned} then\ \parallel R[f] \parallel &= \parallel f(t) - P(t) \parallel \\ &\le \parallel f^2(t) \parallel \{ \int_{t_i}^{t} |p(\tau)\, d\tau| + \int_{t}^{t_{i+1}} |q(\tau) d\tau| \} \end{aligned} \qquad (6.4)$$

for $q(\tau)$ since

$$q(\tau) = \left\{ \begin{array}{l} -h \left\{ (1-\theta) \left[\delta_i(1-\cos\theta)^3 \right.\right. \\ \left. +2\cos\theta(1-\cos\theta)^2\gamma_i \right] \\ \left. -2\cos\theta(1-\cos\theta)^2 \frac{\delta_i}{\pi r_i} \right\} \end{array} \right\} / R \ \le\ 0 \qquad (6.5)$$

and

$$q(t_{i+1}) = \frac{2h\delta_i\cos\theta(1-\cos\theta)^2}{\pi r_i R} \ \ge 0 \qquad (6.6)$$

it is easy to see that the root τ^* of $q(\tau)$ is

$$\tau^* = t_{i+1} - \frac{2h\delta_i \cos\theta(1-\cos\theta)^2}{\pi r_i[\delta_i(1-\cos\theta)^3 + 2\cos\theta(1-\cos\theta)^2\gamma_i]}$$

thus

$$\int_t^{t_{i+1}} |q(\tau)d\tau| = \int_t^{\tau^*} -q(\tau)\,d\tau + \int_{\tau^*}^{t_{i+1}} q(\tau)d\tau = h^2 M_1$$

where

$$M_1 = \frac{z^2 + 4\delta_i^2\cos^2\theta(1-\cos\theta)^4}{2R\pi^2[\delta_i(1-\cos\theta)^3 + 2\cos\theta(1-\cos\theta)^2\gamma_i]} \tag{6.7}$$

where

$$\begin{aligned} z &= 2\cos\theta(1-\cos\theta)^2\delta_i \\ &\quad - \pi r_i\left[\delta_i(1-\cos\theta)^3 + 2\cos\theta(1-\cos\theta)^2\gamma_i\right](1-\theta) \end{aligned}$$

similarly since

$$p(\tau) = q(\tau) \le 0$$

$$p(t_i) = h\left[\theta - \left\{\begin{array}{l} \delta_i(1-\cos\theta)^3 \\ +2\cos\theta(1-\cos\theta)^2\gamma_i \\ -\frac{2\cos\theta(1-\cos\theta)^2\delta_i}{\pi r_i} \end{array}\right\}/R\;\right] \ge 0$$

and the root τ_* of $p(\tau)$ in $[t_i,\ t]$ is

$$\tau_* = t_{i+1} - h\frac{[(1-\theta) - (2\cos\theta(1-\cos\theta)^2\delta_i)/\pi r_i R]}{L}$$

so that

$$\begin{aligned} \int_{t_i}^{t} |p(\tau)|\,d\tau &= \int_{t_i}^{\tau_*} p(\tau)\,d\tau + \int_{\tau_*}^{t} -p(\tau)d\tau \\ &= h^2\ \frac{N_1 + \{(1-\theta) - (2\cos\theta(1-\cos\theta)^2\delta_i)/\pi r_i R\}^2}{1-L} \end{aligned}$$

where

$$N_1 = (1-L)\left[2\theta + \frac{4\cos\theta(1-\cos\theta)^2(2-\theta)}{\pi r_i R} - (1+L)(1+(1-\theta)^2)\right]$$

$$\textit{and } L = ((1-\cos\theta)^3\delta_i + 2\cos\theta(1-\cos\theta)^2\gamma_i)/R$$

$$\| R[f] \| = \| f(t) - P(t) \| \le h^2 \| f^2(t) \| w(\theta, \alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i,\ r_i)$$

where

$$w\,(\theta, \alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i,\ r_i) =\ M_1 + N_1$$

where $w(\theta, \alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i,\ r_i)$ is depending upon $\theta, \alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i$ and r_i.

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ON THE SWIRLING FLOW ANALOGUE OF THE HOWARD'S CONJECTURE IN HYDRODYNAMIC STABILITY

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Abstract. We consider the linear instability problem of ideal incompressible swirling flows to axisymmetric disturbances in the presence of axial velocity. Estimates for growth rate (kc_i) are obtained for two classes of basic flows and as a consequence the swirling flow analogue of the Howard's conjecture, namely, the growth rate tends to zero as the axial wave number tends to infinity, is proved. The first class consists of axial velocity profiles which may be unstable in the absence of swirl, while, the second class consists of flows which may be unstable in the presence of weak swirl.

1. Introduction

Swirling flows are widely studied in the literature. The understanding of mechanisms of their stability (or instability) is of great importance in the design of control strategies for aircraft wakes and for many combustion devices. Furthermore, from a fundamental point of view, swirling flows are present in turbulent flows, and their break-down is a possible mechanism for generating small scales in such flows [5].

The abrupt change in the flow structure of the Swirling flows is known as vortex break-down. Two forms of vortex break-down predominate, one called"axisymmetric" or "bubble-like" and the other called "spiral". Many authors believe that the vortex break-down is a direct consequence of hydrodynamic instability with respect to axisymmetric or non-axisymmetric infinitesimal disturbances [15]. This hydrodynamic stability problem was formulated by [2] in the following way.

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Let (r, θ, z) be a cylindrical polar coordinate system. Consider the motion of an inviscid incompressible homogeneous fluid in an annular region between two infinite coaxial cylinders placed at the radial positions $r = R_1$ and $r = R_2$ with $0 < R_1 < r < R_2 < \infty$. The Euler equations of hydrodynamics allow the steady solution with velocity $(0, V(r), W(r))$, constant density ρ, and pressure $P = \rho \int \frac{V^2(r)}{r} dr$, where the azimuthal velocity $V(r)$ and the axial velocity $W(r)$ are arbitrary functions of the radial coordinate variable r. In considering the stability of the foregoing basic flow, we shall restrict ourselves to infinitesimal axisymmetric perturbations. Let the perturbed state be given by the velocity $(u_r', V + u_\theta', W + u_z')$ and pressure $P + p'$. By considering only axisymmetric infinitesimal normal mode disturbances we take $(u_r', u_\theta', u_z', p') = (u(r), v(r), w(r), p(r))e^{ik(z-ct)}$ where $k > 0$ is the axial wave number and $c = c_r + ic_i$ is the (complex) phase velocity of the disturbances. The stability equation and the associated boundary conditions were derived in terms of the radial component $u(r)$ of the disturbance velocity in [2]. In the above problem a disturbance is unstable if $kc_i > 0$ and kc_i is called the growth rate of an unstable mode. The basic flow is unstable if there is an unstable mode and it is stable if there are no unstable modes. Let us denote the differential operator by D that is $D = d/dr$. Chandrasekhar's attempt in [2] to obtain a sufficient condition for stability in terms of the Rayleigh discriminant $\Phi = \frac{D((rV)^2)}{r^3}$was inconclusive. It is well known that a pure swirl flow is stable if $\Phi \geq 0$ throughout the annulus and physically it means that if the circulation (rV) along streamlines is an increasing function of r then that flow is stable (see, for example, [3]). However in the presence of an axial flow the swirl flow may become unstable due to axial shear. The definitive paper on this problem is by [7]. They introduced the (local) Richardson number $J = \frac{\Phi}{(DW)^2}$ and proved the following results: 1. A sufficient condition for stability is that $J_0 \geq 1/4$ where $J_0 = J_{min}$ with the minimum being taken over $[R_1, R_2]$; 2. For flows with $0 \leq J_0 < 1/4$, the instability region, within which the complex phase velocity c of unstable modes (i.e. with $c_i > 0$) must lie, is the interior of a semicircle, in the upper half of the $c_r - c_i$ plane, whose diameter is the range of the axial velocity W; 3. An estimate for the growth rate of any unstable mode is given by $k^2c_i^2 \leq (\frac{1}{4} - J_0)(DW)^2_{max}$, where the maximum is taken over $[R_1, R_2]$. These results were found as the swirling flow analogues of the corresponding results in the Taylor–Goldstein problem of hydrodynamic stability which deals with the stability of inviscid incompressible density stratified parallel shear flows in the presence of gravity. For the Taylor–Goldstein problem the corresponding results were originally obtained in [12], [6].

By the analogy between swirl in the present problem and density stratification in the Taylor–Goldstein problem a weak swirl characterized by $0 \leq \Phi << 1$ may make the flow unstable. This instability of the flow may be caused not only axial shear but also by a variable azimuthal velocity.

Further progress has been made on the Taylor–Goldstein problem. In particular, the semicircular instability region has been improved to obtain a semi elliptical instability region in [9]. 4. In the swirling flow context, a semi elliptical instability region analogous to the one of [9] has been obtained in [15]. Further it has been pointed out in [15], that the vortex breakdown phenomenon is due to instability of swirling flows with respect to infinitesimal disturbances and that the axial velocity $W(r)$ is also important in this context in addition to the azimuthal velocity $V(r)$.

Now we state the conjecture which is proved in the present paper for two classes of basic swirling flows.

Conjecture: The growth rate kc_i of an unstable axisymmetric normal mode tends to zero as the wave number $k \to \infty$.

This conjecture is the swirling flow analogue of the conjecture made by [6] in the Taylor–Goldstein problem. Howard's conjecture has been proved for a class of parallel shear flows in [1] and for another class of basic parallel flows in [14].

Instabilities of swirling flows are established by computing the growth rates and it is seen from the literature (see, for example, [11] and [8]) that positive growth rates are found only in finite ranges of the wave number k. This suggests that $kc_i = 0$ for large k. As a first step in this direction we have proved that the growth rate $kc_i \to 0$ as $k \to \infty$ for two classes of swirling flows.

The two classes of basic swirling flows for which the conjecture is proved here are the following. The first class consists of flows satisfying the condition that there is an $r_s \in [R_1, R_2]$ such that $\Psi = rD(\frac{DW}{r}) = 0$ at r_s and $\Phi(r_s) = 0$. As mentioned earlier axial shear causes the swirling flow to be unstable. However it is known that a pure axial flow is stable if either $\Psi > 0$ or $\Psi < 0$ throughout the flow domain. Consequently a necessary condition for instability of a pure axial flow is that there is a point $r_s \in (R_1, R_2)$ such that $\Psi(r_s) = 0$. To understand the importance ofΨ on the stability of pure axial flows one can refer to [2] and [13]. The second class consists of flows satisfying the condition $|\Phi| << 1$.

Actually we obtain estimates for growth rate kc_i of unstable modes that depend on the wave number k and by taking the limit as $k \to \infty$ we deduce that $kc_i \to 0$ thus proving the conjecture. We also give examples of basic flows which are members of the above classes. An example of a basic flow that is a member of the first class is constructed here. As an example that is a member of the second class we consider a special case of the Batchelor q- vortex flow the velocity profiles of which has been observed in experiments and whose instability has been studied in many works (see for example, [15]). It is to be noted that due to the importance of the Batchelor q- vortex flow in the vortex break-down problem its instability has been studied to non-axisymmetric disturbances also (see [8] and [11]).

2. Eigenvalue problem

The differential equation which determines the stability to axisymmetric perturbations of an inviscid incompressible flow between two infinite concentric cylinders at $r = R_1, R_2$, where $0 < R_1 < R_2 < \infty$, (cf.[2]) is given by

$$D\left(\frac{D(ru)}{r}\right) - k^2 u - \frac{\Psi u}{(W-c)} + \frac{\Phi u}{(W-c)^2} = 0, \tag{2.1}$$

and the boundary conditions are

$$u(r) = 0 \text{ at } r = R_1, R_2, \tag{2.2}$$

where $\Phi = \frac{1}{r^3} D(r^2 V^2)$, $D = d/dr$ as mentioned earlier. Here the axial velocity $W(r)$ and the azimuthal velocity $V(r)$ are arbitrary twice continuously differentiable functions of r. It should be noted that $u(r)e^{ik(z-ct)}$ is the radial component of the perturbation velocity.

3. Growth Rate Estimates

In this section we shall obtain some estimates for growth rates of unstable disturbances. It is shown as a consequence that the growth rate tends to zero, as the wave number tends to infinity for two classes of basic flows. First we prove the following two lemmas:

Lemma 3.1. *A necessary condition for the existence of a nontrivial solution* (u, c, k^2) *with* $c_i > 0$ *of equations* (2.1) *and* (2.2) *is that the integral relations*

$$\int_{R_1}^{R_2} \frac{|D(ru)|^2}{r} dr + k^2 \int_{R_1}^{R_2} r|u|^2 dr + \int_{R_1}^{R_2} \frac{r\Psi|u|^2}{|W-c|^2}(W-c_r)dr - \int_{R_1}^{R_2} \frac{\Phi r|u|^2((W-c_r)^2-c_i^2)}{|W-c|^4} dr = 0. \quad (3.1)$$

and

$$\int_{R_1}^{R_2} \frac{r\Psi|u|^2}{|W-c|^2} dr - 2\int_{R_1}^{R_2} \frac{\Phi r|u|^2(W-c_r)}{|W-c|^4} dr = 0. \quad (3.2)$$

are true.

Proof. Multiplying equation (2.1) by ru^*(where $*$ stands for complex conjugate) and integrating over $[R_1, R_2]$, we have

$$\int_{R_1}^{R_2} D\left(\frac{D(ru)}{r}\right) ru^* dr - k^2 \int_{R_1}^{R_2} ruu^* dr - \int_{R_1}^{R_2} \frac{\Psi u}{(W-c)} ru^* dr + \int_{R_1}^{R_2} \frac{\Phi u}{(W-c)^2} ru^* dr = 0,$$

Integrating it by parts once and using boundary conditions (2.2), we have

$$\int_{R_1}^{R_2} \frac{|D(ru)|^2}{r} dr + k^2 \int_{R_1}^{R_2} r|u|^2 dr + \int_{R_1}^{R_2} \frac{r\Psi|u|^2}{(W-c)} dr - \int_{R_1}^{R_2} \frac{\Phi r|u|^2}{(W-c)^2} dr = 0. \quad (3.3)$$

Equating the real and imaginary parts of both sides of equation (3.3) and cancelling $c_i(> 0)$ throughout from the imaginary parts, we get

$$\int_{R_1}^{R_2} \frac{|D(ru)|^2}{r} dr + k^2 \int_{R_1}^{R_2} r|u|^2 dr + \int_{R_1}^{R_2} \frac{r\Psi|u|^2}{|W-c|^2}(W-c_r)dr - \int_{R_1}^{R_2} \frac{\Phi r|u|^2((W-c_r)^2-c_i^2)}{|W-c|^4} dr = 0.$$

and

$$\int_{R_1}^{R_2} \frac{r\Psi|u|^2}{|W-c|^2} dr - 2\int_{R_1}^{R_2} \frac{\Phi r|u|^2(W-c_r)}{|W-c|^4} dr = 0,$$

which completes the proof of the Lemma. □

Remark 3.2. For a pure axial flow $\Phi = 0$ and from (3.2) one can deduce that $\Psi = 0$ atleast once is necessary for instability.

Remark 3.3. We can drop the first term of (3.1) to get

$$k^2 \int_{R_1}^{R_2} r|u|^2 dr \leq \int_{R_1}^{R_2} \frac{r\Psi|u|^2}{|W-c|^2}(W-c_r)dr + \int_{R_1}^{R_2} \frac{\Phi r|u|^2}{|W-c|^2} dr.$$

From this one gets the estimate

$$k^2c_i^2 \leq |\Psi|_{max}(b-a) + \Phi_{max},$$

where $a = W_{min}$ and $b = W_{max}$.

However it may be noted that we cannot conclude the validity of this conjecture from their estimate.

Lemma 3.4. *A necessary condition for the existence of a nontrivial solution* (u, c, k^2) *with* $c_i > 0$ *of equations* (2.1) *and* (2.2) *is that the integral relations*

$$\int_{R_1}^{R_2} |D\left(\frac{D(ru)}{r}\right)|^2 rdr + k^2\int_{R_1}^{R_2} \frac{|D(ru)|^2}{r}dr - k^2\int_{R_1}^{R_2} \frac{r\Psi(W-c_r)|u|^2}{|W-c|^2}dr$$
$$-\int_{R_1}^{R_2} \frac{\Psi^2 r|u|^2}{|W-c|^2}dr + 2\int_{R_1}^{R_2} \frac{r\Psi\Phi|u|^2(W-c_r)}{|W-c|^4}dr$$
$$+k^2\int_{R_1}^{R_2} \frac{r\Phi|u|^2((W-c_r)^2-c_i^2)}{|W-c|^4}dr - \int_{R_1}^{R_2} \frac{\Phi^2 r|u|^2}{|W-c|^4}dr = 0 \tag{3.4}$$

and

$$\int_{R_1}^{R_2} \frac{r\Psi(W-c_r)|u|^2}{|W-c|^2}dr - 2\int_{R_1}^{R_2} \frac{r\Phi|u|^2(W-c_r)}{|W-c|^4}dr = 0 \tag{3.5}$$

are true.

Proof. Multiplying equation (2.1) by $D(\frac{D(ru^*)}{r}).r$ and integrating over $[R_1, R_2]$, we have

$$\int_{R_1}^{R_2} \left|D\left(\frac{D(ru)}{r}\right)\right|^2 rdr - k^2\int_{R_1}^{R_2} D(\frac{D(ru^*)}{r})rudr - \int_{R_1}^{R_2} \frac{D(\frac{D(ru^*)}{r})r\Psi u}{(W-c)}dr+$$
$$\int_{R_1}^{R_2} \frac{D(\frac{D(ru^*)}{r})r\Phi u}{(W-c)^2}dr = 0. \tag{3.6}$$

Taking complex conjugate of both sides of equation (2.1) gives

$$D\left(\frac{D(ru^*)}{r}\right) = k^2u^* + \frac{\Psi u^*}{(W-c)} - \frac{\Phi u^*}{(W-c)^2}, \tag{3.7}$$

Substituting (3.7) in (3.6) and integrating the second term by parts once,

$$\int_{R_1}^{R_2} |D(\frac{D(ru)}{r})|^2 rdr + k^2\int_{R_1}^{R_2} \frac{|D(ru)|^2}{r}dr - \int_{R_1}^{R_2} \frac{r\Psi u}{(W-c)}(k^2u^* + \frac{\Psi u^*}{(W-c^*)}$$
$$-\frac{\Phi u^*}{(W-c^*)^2})dr + \int_{R_1}^{R_2} \frac{r\Phi u}{(W-c)^2}(k^2u^* + \frac{\Psi u^*}{(W-c^*)} - \frac{\Phi u^*}{(W-c^*)^2})dr = 0.$$

The above equation can be written as,

$$\int_{R_1}^{R_2} |D(\frac{D(ru)}{r})|^2 r dr + k^2 \int_{R_1}^{R_2} \frac{|D(ru)|^2}{r} dr - k^2 \int_{R_1}^{R_2} \frac{r\Psi|u|^2}{(W-c)} dr$$
$$- \int_{R_1}^{R_2} \frac{\Psi^2 r|u|^2}{|W-c|^2} dr + \int_{R_1}^{R_2} \frac{r\Psi\Phi|u|^2}{(W-c)(W-c^*)^2} dr + k^2 \int_{R_1}^{R_2} \frac{r\Phi|u|^2}{(W-c)^2} dr$$
$$+ \int_{R_1}^{R_2} \frac{r\Psi\Phi|u|^2}{(W-c^*)(W-c)^2} dr - \int_{R_1}^{R_2} \frac{\Phi^2 r|u|^2}{|W-c|^4} dr = 0. \tag{3.8}$$

Equating the real and imaginary parts of both sides of equation (3.8) and cancelling $k^2 c_i(>0)$ throughout from the imaginary part gives,

$$\int_{R_1}^{R_2} \left| D\left(\frac{D(ru)}{r}\right)\right|^2 r dr + k^2 \int_{R_1}^{R_2} \frac{|D(ru)|^2}{r} dr - k^2 \int_{R_1}^{R_2} \frac{r\Psi(W-c_r)|u|^2}{|W-c|^2} dr$$
$$- \int_{R_1}^{R_2} \frac{\Psi^2 r|u|^2}{|W-c|^2} dr + 2\int_{R_1}^{R_2} \frac{r\Psi\Phi|u|^2(W-c_r)}{|W-c|^4} dr + k^2 \int_{R_1}^{R_2} \frac{r\Phi|u|^2((W-c_r)^2-c_i^2)}{|W-c|^4} dr$$
$$- \int_{R_1}^{R_2} \frac{\Phi^2 r|u|^2}{|W-c|^4} dr = 0$$

and

$$\int_{R_1}^{R_2} \frac{r\Psi(W-c_r)|u|^2}{|W-c|^2} dr - 2\int_{R_1}^{R_2} \frac{r\Phi|u|^2(W-c_r)}{|W-c|^4} dr = 0.$$

which completes the proof of the Lemma. □

Remark 3.5. From (3.4) one is not able to get any estimate for the growth rate. However comparison of (3.1) and (3.4) shows that two terms, namely, the third term and the fifth term of (3.4) are $(-k^2)$ times the third and fourth terms of (3.1). This fact is used in the proof of the following theorem.

Theorem 3.6. *If $r = r_s$ be such that $R_1 < r_s < R_2$ with $\Psi(r_s) = 0, \Phi(r_s) = 0$, and $\lambda^2\Psi^2$ is bounded in $[R_1, R_2]$ then a necessary condition for the existence of a nontrivial solution (u, c, k^2) of (2.1) and (2.2) with $c_i > 0$ and $c_r = W_s = W(r_s)$ is that*

$$k^2 c_i \leq \max_{R_1 \leq r \leq R_2} \{\lambda|\Psi|\}, \tag{3.9}$$

$$\text{and } \lambda = \left[\left(1 - \frac{\Phi}{\Psi(W-W_s)}\right)^2 + 1\right]^{1/2}. \tag{3.10}$$

Proof. Multiplying equation (3.1) by k^2 and adding the resultant equation to (3.4), we have

$$\int_{R_1}^{R_2} |D\left(\frac{D(ru)}{r}\right)|^2 r dr + 2k^2 \int_{R_1}^{R_2} \frac{|D(ru)|^2}{r} dr + k^4 \int_{R_1}^{R_2} r|u|^2 dr - \int_{R_1}^{R_2} \frac{\Psi^2 r|u|^2}{|W-c|^2} dr + 2\int_{R_1}^{R_2} \frac{r\Psi\Phi|u|^2(W-c_r)}{|W-c|^4} dr - \int_{R_1}^{R_2} \frac{\Phi^2 r|u|^2}{|W-c|^4} dr = 0. \tag{3.11}$$

The first two terms of the above equation are non-negative and so they can be dropped to get,

$$k^4 \int_{R_1}^{R_2} r|u|^2 dr - \int_{R_1}^{R_2} \frac{\Psi^2 r|u|^2}{|W-c|^2} dr \quad +2\int_{R_1}^{R_2} \frac{r\Psi\Phi|u|^2(W-c_r)}{|W-c|^4} dr - \int_{R_1}^{R_2} \frac{\Phi^2 r|u|^2}{|W-c|^4} dr \leq 0. \tag{3.12}$$

This relation can be rewritten as

$$k^4 \int_{R_1}^{R_2} r|u|^2 dr - \int_{R_1}^{R_2} \left[\Psi^2((W-c_r)^2 + c_i^2) - 2\Psi\Phi(W-c_r) + \Phi^2\right] \frac{|u|^2}{|W-c|^4} r dr \leq 0, \tag{3.13}$$

$$i.e., k^4 \int_{R_1}^{R_2} |u|^2 r dr - \int_{R_1}^{R_2} \left[(\Psi(W-c_r) - \Phi)^2 + \Psi^2 c_i^2\right] \frac{r|u|^2}{|W-c|^4} dr \leq 0;$$

Now we take $c_r = W(r_s)$ and use the hypotheses of the theorem to write the above relation as

$$k^4 \int_{R_1}^{R_2} |u|^2 r dr - \int_{R_1}^{R_2} \frac{\left[\Psi^2(W-c_r)^2(1-\frac{\Phi}{\Psi(W-c_r)})^2 + \Psi^2 c_i^2\right]}{|W-c|^4} r|u|^2 dr \leq 0.$$

Since $(W-c_r)^2 \leq |W-c|^2$ and $c_i^2 \leq |W-c|^2$ this yields

$$k^4 \int_{R_1}^{R_2} r|u|^2 dr - \int_{R_1}^{R_2} \Psi^2 \left[1 + \left(1 - \frac{\Phi}{\Psi(W-c_r)}\right)^2\right] \frac{r|u|^2}{|W-c|^2} dr \leq 0;$$

$$i.e., k^4 \int_{R_1}^{R_2} |u|^2 r dr - \int_{R_1}^{R_2} \frac{\lambda^2\Psi^2|u|^2}{c_i^2} r dr \leq 0;$$

$$i.e., k^4 - \frac{\Psi^2\lambda^2}{c_i^2} \leq 0, \tag{3.14}$$

where the value of λ is given in (3.10). From (3.14), it follows that

$$k^4 c_i^2 \leq [\Psi^2 \lambda^2]_{max}.$$

So an upper bound for the growth rate of an arbitrary unstable wave is given by (3.9), which completes the proof of the theorem. $\square$

Theorem 3.7. *For basic flows satisfying the condition of boundedness of the quantity* $\lambda^2\Psi^2$, *we have* $\lim_{k\to\infty} kc_i = 0$.

Proof. Proof follows from (3.9). $\square$

Now we shall give an example of basic flow that satisfies the condition of Theorem 3.1.

Example 3.8.

Consider $W(r) = tanh^2\left[(r - \frac{R_1+R_2}{2})^2\right]$ with $R_1 = 1, R_2 = 2$.
Therefore $W(r) = tanh^2\left[r - \frac{3}{2}\right]^2$, where $r \in [1, 2]$
and r_s is the point at which $\Psi = 0$.
We shall check that $\lambda^2\Psi^2$ is bounded for this flow.
It is seen that $\Psi = 0$ at $r = r_s = 3/2$.
And also consider $V = tanh(r - \frac{R_1+R_2}{2})$ with $R_1 = 1, R_2 = 2$.
So $V(r) = tanh(r - \frac{3}{2})$,
$\Phi = \frac{1}{r^3}D(r^2V^2) = 0$ at $r = r_s = 3/2$.
To check that $\lambda^2\Psi^2$ is bounded, it is seen that

$$\begin{aligned}\lambda^2\Psi^2 &= \left(1 + (1 - \frac{\Phi}{\Psi(W - W_s)})^2\right)\Psi^2;\\ &= (2 + \frac{\Phi^2}{\Psi^2(W-W_s)^2} - \frac{2\Phi}{\Psi(W-W_s)})\Psi^2;\\ &= \frac{2\Psi^2(W-W_s)^2 + \Phi^2 - 2\Phi\Psi(W-W_s)}{(W-W_s)^2}. \end{aligned} \tag{3.15}$$

The numerator is bounded in $[R_1, R_2]$. It is found that the denominator is zero of order 2 at $r = r_s$ and nowhere else. But the numerator is also zero at $r = r_s$ of order two or more. Therefore $\lambda^2\Psi^2$ is bounded for this example. The conditions of Theorem 3.1 are satisfied by this example.

Next we consider the basic flows with weak swirl to prove the result of Theorem 3.2.

Theorem 3.9. *If* (u, c, k^2) *is a non-trivial solution of equations* (2.1) *and* (2.2) *with* $c_i > 0$ *and* $|\Phi| << 1$, *then* $kc_i \to 0$ *as* $k \to \infty$.

Proof. Now consider

$$\left|1 - \frac{\Phi}{\Psi(W-c)}\right|^2 = \left(1 - \frac{\Phi}{\Psi(W-c)}\right)\left(1 - \frac{\Phi}{\Psi(W-c^*)}\right);$$
$$= 1 - \frac{2\Phi(W-c_r)}{\Psi|W-c|^2} + \frac{\Phi^2}{\Psi^2|W-c|^2}. \tag{3.16}$$

Under the weak stratification, the last term on the right hand side is neglected and using the equation (3.13), we have

$$k^4 \int_{R_1}^{R_2} r|u|^2 dr - \int_{R_1}^{R_2} \frac{\Psi^2 r|u|^2}{c_i^2}\left[1 - \frac{2\Phi(W-c_r)}{\Psi|W-c|^2}\right]^2 dr \leq 0;$$
$$i.e.,\ k^4 \int_{R_1}^{R_2} r|u|^2 dr - \int_{R_1}^{R_2} \frac{\Psi^2 r|u|^2}{c_i^2} dr + 2\int_{R_1}^{R_2} \frac{\Psi^2 r|u|^2}{c_i^2}\frac{\Phi(W-c_r)}{\Psi|W-c|^2} dr \leq 0. \tag{3.17}$$

Since $|W-c|^2 = (W-c_r)^2 + c_i^2 \geq 2c_i(W-c_r)$, we have from (3.17)

$$k^4 \int_{R_1}^{R_2} r|u|^2 dr - \int_{R_1}^{R_2} \frac{\Psi^2 r|u|^2}{c_i^2} dr + \int_{R_1}^{R_2} \frac{\Psi\Phi r|u|^2}{c_i^3} dr \leq 0. \tag{3.18}$$

Since $\Psi \geq -|\Psi|$, we have

$$k^4 \int_{R_1}^{R_2} r|u|^2 dr - \int_{R_1}^{R_2} c_i \frac{\Psi^2 r|u|^2}{c_i^3} dr + \int_{R_1}^{R_2} \frac{\Psi\Phi r|u|^2}{c_i^3} dr \leq 0. \tag{3.19}$$

The integrand of the 2^{nd} term should be negative and since $c_i \leq (\frac{b-a}{2})$ by the semicircle theorem of [7], where $a = W_{min}$ and $b = W_{max}$, in which minimum and maximum are taken over $[R_1, R_2]$, we have the following inequality,

$$k^4 - \frac{(b-a)\Psi^2}{2c_i^3} - \frac{\Phi|\Psi|}{c_i^3} \leq 0;$$
$$i.e.,\ k^4 c_i^3 \leq \Psi_{max}^2\left(\frac{b-a}{2}\right) + \Phi_{max}|\Psi_{max}|;$$
$$i.e.,\ k^3 c_i^3 \leq \frac{\Psi_{max}^2(\frac{b-a}{2})}{k} + \frac{\Phi_{max}|\Psi_{max}|}{k}. \tag{3.20}$$

This implies that $kc_i \to 0$ as $k \to \infty$ and this completes the proof of the theorem. □

Example 3.10.

Consider the basic flow with

$$W(r) = W_0 e^{-r^2}, V(r) = \frac{q}{r}(1 - e^{-r^2}) \tag{3.21}$$

with $R_1 = 1, R_2 = 2$.
Here W_0 is a constant and the constant q is the swirl parameter. It may be noticed here that this is a special case of the Batchelor q-vortex flow whose instability has been studied in many works (see, for example [15]).
Then $\Psi = -2r^2 W_0 e^{-r^2}$ and $\Phi = \frac{4q^2}{r^2} e^{-r^2}(1 - e^{-r^2})$.
It is seen that $|\Phi| << 1$ if $|q| << 1$.

4. Concluding Remarks

In this paper we have studied the stability of inviscid flows between coaxial cylinders when an axial pressure gradient is present. We consider infinitesimal axisymmetric disturbances and prove that the growth rate $kc_i \to 0$ as the wave number $k \to \infty$ for two classes of basic flows. We have also given examples of basic flows that belong to these classes.

It is desirable to extend the results of the present paper to the problem of instablility of variable density swirling flows that has been studied in the large wave number limit in [4].

It has been observed in [10] that the break-down occurs in flows intermediate between weakly swirling flows, which exhibit no flow reversal in the axial direction and rapidly swirling flows with columnar flow reversals. Since the discriminant Φ is the measure of swirl we should try to prove the analogue of our results for flows without any condition on Φ as we have supposed in our Theorem 3.3.

For flows with constant pressure gradient the basic flow becomes a pure axial flow and this has been studied in some detail in [2]. In this context a stronger result, namely, there is a critical wave number k_c such that any disturbance with $k > k_c$ is stable has been proved recently in [13]. Whether such a result is true in the swirling flow context is unknown.

The stability analysis presented in this paper is restricted to the case of axisymmetric perturbations. Though this is an important restriction, the study of the axisymmetric case has an interest of its own, and leads to a physical understanding of one important type of instability to which non-parallel flows in general are subjects. A stability equation for constant density inviscid incompressible swirling flows with respect to general disturbances, that is, for disturbances of the form {the function of r}$e^{i(-\omega t+m\theta+kz)}$, where ω is the frequency, m is the azimuthal wave number and k is the axial wave number was derived in [7]. A large azimuthal wave number asymptotic analysis of stability based on the equation of [7] was developed in [10]. This analysis was extended to variable density swirling flows in [4]. Moreover a larger class of basic swirling flows was considered in [4]. Interestingly, growth rates for axisymmetric disturbances with large axial wave number was also presented in [4]. It is found that the growth rate is not zero even in the large axial wave number limit.

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INTERIOR VERTICES AND BOUNDARY VERTICES USING GEODESIC AND DETOUR DISTANCES

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Abstract. The first part of this article deals with interior vertices and boundary vertices using standard distance in graphs. A characterization for a vertex to be an interior vertex and boundary vertex in a tree is obtained. The second part of the article concentrates on detour distance in graphs. It is proved that for a detour self centered graph $G : (V, E)$, $e_D(v) = n - 1$ (Conjecture 2.8 of [3]). Then the concepts of detour boundary vertex and detour interior vertex in a graph are introduced and some properties are established.

1. Introduction

The concept of interior vertex and boundary vertex is defined and studied in [2]. In this paper our attention is on interior vertex and boundary vertex and study the properties using spanning tree. The concept of detour distance was introduced by Gary Chartrand and Ping Zhang and further studied by Gary Chartrand, H. Escuadro and Ping Zhang in [4]. In this paper we study more about detour eccentric subgraph, detour periphery, and detour self centered graph. We observe that Conjecture 2.8 by Gary Chartrand and Ping Zhang [3] is true and have obtained a proof for the same. The concepts of detour boundary vertex and detour interior vertex in a graph are introduced, and some properties of detour boundary vertices, detour interior vertices and complete vertices in a graph are studied.

2. Preliminaries

The following basic concepts are taken from [1, 2].

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Let $G:(V,E)$ be a graph. The number of edges incident with a vertex v is called the degree of v in a graph G. A vertex of degree one is called a pendant vertex of the graph. A graph H is called a subgraph of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If the subgraph of G has same vertex set as G, then it is a spanning subgraph of G. A spanning subgraph H of a connected graph G such that H is a tree is called a spanning tree of G. A vertex v in a connected graph G is a cut vertex of G if $G-v$ is disconnected. Let G be a non-trivial connected graph. The standard distance $d(u,v)$ between two vertices u and v of G is the length of a $u-v$ shortest path in G. A $u-v$ path of length $d(u,v)$ is called $u-v$ geodesic. If u and v are vertices in G and w is a neighbor of u then $|d(u,v)-d(w,v)| \leq 1$. For a vertex u in a connected graph G, the eccentricity $e(u)$ of u is the distance between u and a vertex farthest from u in G. The minimum eccentricity among the vertices of G is called radius of G, denoted by $rad(G)$. A vertex v is a central vertex if $e(v)=rad(G)$. The subgraph induced by the central vertices of G is the center $C(G)$ of G. The maximum eccentricity among the vertices of G is called diameter of G, denoted by $diam(G)$. A vertex u is called neighbor of v if u is adjacent to v. A graph G is called self centered if $rad(G)=diam(G)$.

Example 2.1. Consider the graph given in Figure 1.

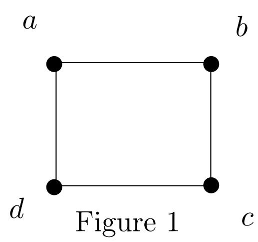

Figure 1

Here $e(a)=2$, $e(b)=2$, $e(c)=2$, $e(d)=2$ and $rad(G)=diam(G)$. Hence the graph is self centered.

A vertex v in a connected graph G is a boundary vertex of a vertex u if $d(u,w) \leq d(u,v)$ for each neighbor w of v; while a vertex v is a boundary vertex of the graph G if v is a boundary vertex of some vertex of G. Subgraph of G induced by its boundary vertices is defined to be the boundary of G denoted by $\partial(G)$ [2]. A vertex y distinct from x and z is said to lie between x and z if $d(x,z) = d(x,y) + d(y,z)$. A vertex v is an interior vertex of G if for every vertex u distinct from v, there exist a vertex w such that v lies between u and w. The interior Int(G) of G is the subgraph of G induced by interior vertices. A vertex v in a graph G is a complete vertex if the subgraph induced by its neighbors form a complete graph. Throughout, we assume that G is a connected graph.

3. Interior vertices boundary vertices and spanning trees

In this section a sufficient condition for a vertex to be an interior vertex in a graph and a characterization for a vertex to be an interior vertex of a tree are obtained. The relationship between cut vertices and interior vertices is also discussed. A sufficient condition for a vertex to be a boundary vertex is obtained. Then a characterization for a vertex to be a boundary vertex in tree is established.

Interior vertices

Theorem 3.1. *If v is an internal vertex of every spanning tree of a connected graph $G = (V, E)$ with atleast 3 nodes, then v is an interior vertex of G.*

Proof. Assume that v in an internal vertex of every spanning tree T of $G = (V, E)$. Then for every vertex distinct from v there exists a vertex w in T such that v lies between u and w. Hence v is an interior vertex of G.

Remark 3.2. The converse of Theorem 3.1 need not be true in general.

Example 3.3. Consider the graph given in Figure 2 and its spanning tree in Figure 3.

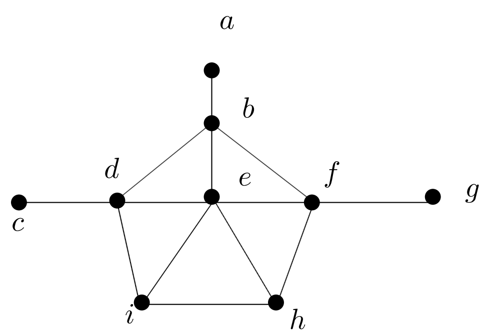

Figure 2

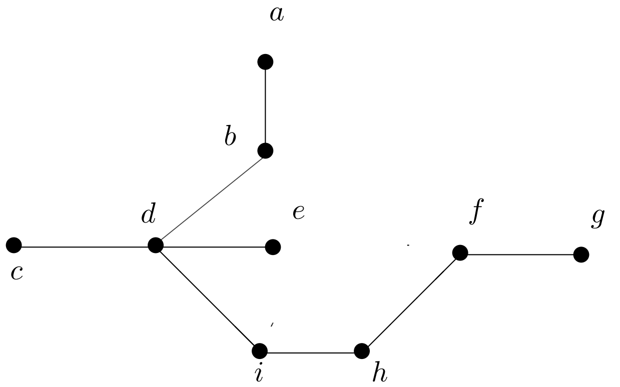

Figure 3

In Figure 2, e is an interior vertex.

Figure 3 is a spanning tree T of the graph G in Figure 2 in which e is a pendant vertex of T.

The converse of Theorem 3.1 is true only in trees as in the following theorem.

Theorem 3.4. *A vertex is an interior vertex of a tree T if and only if it is an internal vertex of a tree.*

Proof. Assume v is an interior vertex of tree T. Then v is not a pendant vertex of T (by definition of interior vertex). Thus v is an internal vertex of T. Converse follows from Theorem 3.1.

Theorem 3.5. *If u is a cut vertex of a connected graph $G = (V, E)$, then u is an interior vertex of G .*

Proof. Assume u is a cut vertex of graph G. Then u is an internal vertex of every spanning tree of G. Then (by Theorem 3.1) u is an interior vertex of G.

Remark 3.6. The converse of Theorem 3.5 need not be true. That is an interior vertex of a graph G need not be a cut vertex.

Example 3.7. Consider the graph given in Figure 2. Here vertex e is an interior vertex but not a cut vertex.

Remark 3.8. $Int(G)$ of a tree is obtained by deleting all pendant vertices of that tree.

Remark 3.9. Note that if u is an interior vertex of a graph G, then it can be cut vertex or central vertex or neither of these.

Example 3.10. Consider the graph given in Figure 4.

In Figure 4, the interior vertices are b, d, e and f, where d is a cut vertex, b and e are central vertices, and f is neither a cut vertex nor a central vertex.

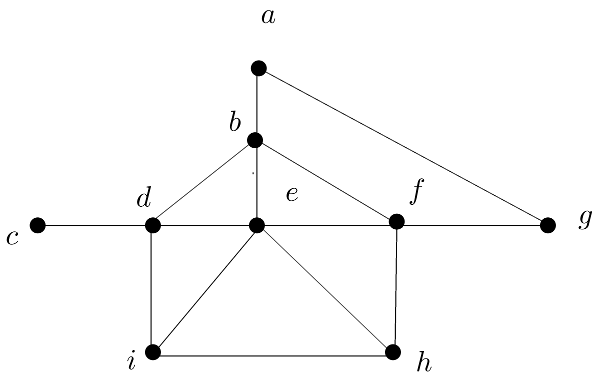

Figure 4

Boundary vertex

Theorem 3.11. *If u is a pendant vertex of every spanning tree of a connected graph $G = (V, E)$, then u is a boundary vertex of G.*

Proof. Assume that u is a pendant vertex of every spanning tree T of $G = (V, E)$. Then u cannot be a cut vertex of G (by Theorem 8 of [5]). Also no cut vertex can be boundary vertex [1]. Hence u is a boundary vertex of G.

Remark 3.12. The converse of Theorem 3.11 need not be true in general.

Example 3.13. Consider the graph given in Figure 5 and Figure 6.

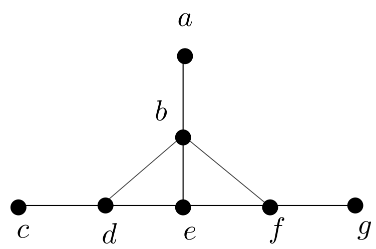

Figure 5

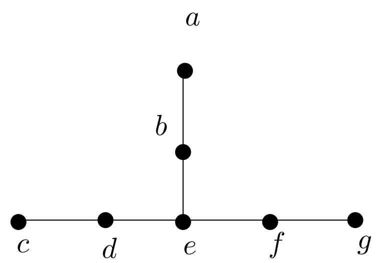

Figure 6

In Figure 5, e is a boundary vertex.
Figure 6 is a spanning tree T of the graph G in Figure 5 in which e is an internal vertex .

The converse of Theorem 3.11 is true only in trees.

Theorem 3.14. *A vertex is a boundary vertex of a tree if and only if it is a pendant vertex.*

Proof. Let G be a tree and u is a boundary vertex of some vertex v in G. In a tree all internal vertices are interior vertices (Theorem 3.4). Also a vertex is a boundary vertex of a graph G if and only if it is not an interior vertex of G [1]. Hence u should be pendant vertex.

Remark 3.15. The boundary of G denoted by $\partial(G)$ of a tree is set of all pendant vertices of that tree.

4. Detour self centered graph

The following basic concepts are taken from [1, 3, 4].

The length of a longest $u-v$ path between two vertices u and v in a connected graph G is called detour distance from u to v, denoted by $D(u,v)$. A $u-v$ path of length $D(u,v)$ is a $u-v$ detour.

The detour eccentricity, $e_D(u)$ of a vertex u is the detour distance from u to a vertex farthest from u. The detour eccentric vertex of u in G is a vertex v such that $e_D(u) = D(u,v)$. Let u_D^* denote set of all detour eccentric vertices of u. A connected graph $G : (V,E)$ is a detour eccentric graph if every vertex of G is a detour eccentric vertex. Let $G : (V,E)$ be a connected graph. The detour eccentric graph of G, $Ecc_D(G)$, is the subgraph of G induced by the set of detour eccentric vertices of G. The detour radius of G, $rad_D(G)$ is the minimum detour eccentricity among the vertices of G. A vertex u in G is a detour central vertex if, $e_D(u) = rad_D(G)$. The subgraph of G induced by the detour central vertices is called detour centre of G, denoted by $C_D(G)$. If every vertex of G is detour central vertex then $C_D(G) = G$, and G is called detour self centered. The detour diameter of G, $diam_D(G)$ is the maximum detour eccentricity among the vertices of G. Note that for detour self centered graph $rad_D(G) = diam_D(G)$. A vertex u in a connected graph G is called detour peripheral vertex if $e_D(u) = diam_D(G)$. A $u-v$ detour whose length is equal to detour diameter is called detour peripheral path. The subgraph induced by its detour peripheral vertices is the detour periphery, $per_D(G)$.

Example 4.1. Consider the graph given in Figure 7.

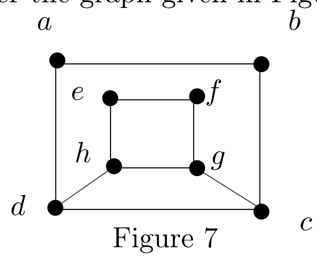

Figure 7

In Figure 7, $D(a,d) = 7$,which is detour diameter of G. Hence a, b, c, g, f, e, h, d is a detour peripheral path. All vertices of G have eccentricity 7, and therefore $rad_D(G) = diam_D(G)$. Hence the graph is detour self centered.

With respect to standard distance note that a necessary condition for a self centered graph is that each vertex is eccentric, which is not sufficient.

But we observe that in detour distance it is sufficient also, which is discussed as follows.

Theorem 4.2. *A connected graph $G : (V, E)$ is detour self centered graph if and only if each vertex of G is detour eccentric.*

Proof. Assume $G : (V, E)$ is connected detour self centered graph and let v be any vertex of G. Let u be any detour eccentric vertex of v, that is $u \in v^*_D$. Then $e_D(v) = D(u, v)$ and G being detour self centered graph $e_D(u) = e_D(v) = D(u, v)$, which shows that $v \in u^*_D$, and v is detour eccentric.

Conversely assume that each vertex of G is detour eccentric. To prove that G is detour self centered graph. Assume to the contrary, that G is not detour self centered, ie, $rad_D(G) \neq diam_D(G)$. Let y be a vertex in G such that $e_D(y) = diam_D(G)$, and let $z \in y^*_D$. Let P be a $y - z$ detour in G. Then there must exists a vertex w on P such that w is not detour eccentric vertex of any vertex of P. Also w is not a detour eccentric vertex of any other vertex. Otherwise if w is a detour eccentric vertex of any other vertex u (say), ie, $w \in u^*_D$. Then we can extend $u - w$ detour to longer path(to y or to z or to both), which is a contradiction to $w \in u^*_D$. $\therefore$ $rad_D(G) = diam_D(G)$ and G is detour self centered graph.

Remark 4.3. We observe that Conjecture 2.8 by Gary Chartrand and Ping Zhang [3] is true and obtained a proof for the same in Theorem 4.4.

Theorem 4.4. *For a detour self centered graph $G : (V, E)$ of order n, $e_D(v) = n - 1$, $\forall\ v \in V(G)$.*

Proof. Assume $G : (V, E)$ is detour self centered. To prove $e_D(v) = n - 1$, $v \in V$. Assume to the contrary that $e_D(v) = k < n - 1$, $v \in V$.
Then we claim that there exists a vertex x in G which is a common vertex of all detour peripheral paths. If not let P_1 and P_2 be two detour peripheral paths such that P_1 and P_2 have no common vertex. Let $y \in V(P_1)$ and $z \in V(P_2)$. Since G is connected there exists a path from z to y. Since P_1 and P_2 are disjoint peripheral paths, there exist atleast one path P in G longer than P_1 and P_2. Hence there exists vertices on P with detour eccentricity greater than k, which is not possible. Hence our claim is true.
Since x is on every detour peripheral paths, $e_D(x) < k$, which is a contradiction to our assumption that G is detour self centered.

5. Detour periphery and Detour eccentric sub graph

In this section a characterization of detour eccentric vertex is obtained. Also, note that the proof of the Theorem 5.1 and Theorem 5.2 follows from the proof of Theorem 4.4.

Theorem 5.1. *Let $G : (V, E)$ be a connected non trivial graph on n vertices. Then $Per_D(G) = G$ if and only if every vertex of G has detour eccentricity $n - 1$.*

Theorem 5.2. *Let $G : (V, E)$ be a connected non trivial graph on n vertices. Then $Ecc_D(G) = G$ if and only if every vertex of G has detour eccentricity $n - 1$.*

Remark 5.3. With respect to standard distance every peripheral vertex is an eccentric vertex, but not conversely. But in detour distance we observe that converse is also true.

Theorem 5.4. *Let $G : (V, E)$ be a connected non trivial graph. Then v is detour eccentric vertex if and only if v is detour peripheral vertex.*

Proof. If $G = (V, E)$ is any graph on 2 or 3 vertices then the result is true. Hence assume the graph has atleast 4 vertices. Let v be a detour eccentric vertex of G and let $v \in u^*_D$. Let y and z be two detour peripheral vertices, that is $D(y, z) = k = diam_D(G)$. Let P be any $y - z$ detour in G and Q be any $u - v$ detour in G. To prove $e_D(u) = diam_D(G)$. Assume on the contrary that $e_D(u)= l < diam_D(G)$.

Then we have two cases.

Case.1

v is an internal vertex (deg(v)>1) of G. Since G is connected, there exists at least one path between every pair of vertices. Therefore, there exists connection from v to y and from v to z also. So we can extend $u - v$ detour to y or z, which contradicts that v is a detour eccentric vertex of u. Hence $e_D(u) = diam_D(G)$. Thus v is a detour peripheral vertex.

Case.2

v is not an internal vertex (deg(v)=1) of G. Let w be the only vertex adjacent to v. Then w should belong to Q. Now graph being connected w is connected to some vertex, say $w^|$ of P.

Then either $w^|$ is not in Q or $w^|$ is a common vertex of both P and Q.

In both cases the path from u to z or to y through w and $w^|$ is longer than

Q, which contradicts that v is a detour eccentric vertex of u. Hence $e_D(u)$ $= diam_D(G)$. Thus v is a detour peripheral vertex.

Conversely, assume v is a detour peripheral vertex of G. Then there exists at least one more detour peripheral vertex (say u) and v is detour eccentric vertex of u.

6. Detour boundary vertex and Detour interior vertex of a graph

In this section detour boundary vertex, detour boundary, detour interior vertex and detour interior of a graph are defined. A result connecting detour interior vertex and detour boundary vertex is obtained. A necessary and sufficient condition for a vertex to be detour boundary vertex and detour interior vertex is obtained.

Definition 6.1. A vertex v in a connected graph G:(V,E) is a detour boundary vertex of a vertex u if $D(u, v) \geq D(u, w)$ for each neighbor w of v, while a vertex v is a detour boundary vertex of a graph G if v is a detour boundary vertex of some vertex of G. Let u_D^b denote set of all detour boundary vertices of u.

Definition 6.2. The subgraph of G induced by its detour boundary vertices is called the detour boundary of G, denoted by $\partial_D(G)$. A graph H is said to be detour boundary graph if $H = \partial_D(G)$ for some connected graph G. A connected graph is detour self boundary graph if $G = \partial_D(G)$.

Definition 6.3. Any vertex y in a connected graph $G : (V, E)$ is said to lie between two other vertices say x and z (both different from y) with respect to detour distance if $D(x, z) = D(x, y) + D(y, z)$.

Definition 6.4. A vertex v is a detour interior vertex of a connected graph $G : (V, E)$ if for every vertex u distinct from v, there exists a vertex w such that v lies between u and w .

Definition 6.5. The detour interior of G, $Int_D(G)$ is the subgraph of G induced by its detour interior vertices.

Example 6.6. Consider the graph given in Figure 2.

Here $a_D^b = \{c, e, i, h, g\}$, $b_D^b = \{a, c, e, i, h, g\}$, $c_D^b = \{a, e, i, h, g\}$, $d_D^b =$ $\{a, c, e, i, h, g\}$, $e_D^b = \{a, c, e, i, h, g\}$, $f_D^b = \{a, c, e, i, h, g\}$, $g_D^b = \{a, c, e, i, h\}$,

$h_D^b = \{a, c, e, i, g\}$, $i_D^b = \{a, c, e, h, g\}$.
Hence the detour boundary vertices of G are a, c, e, g, h and i.

Example 6.7. Consider the graph given in Figure 2.

The detour interior vertices are d, b, and f. Hence $Int_D(G)$ is given in Figure 8.

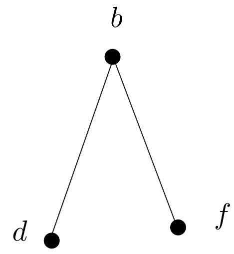

Figure 8

Theorem 6.8. *Let $G : (V, E)$ be a connected graph. A vertex v is a detour boundary vertex of G if and only if v is not a detour interior vertex of G.*

Proof. Let v be a detour boundary vertex of a connected graph $G : (V, E)$, say v is a detour boundary vertex of the vertex u. Assume, to the contrary, that v is a detour interior vertex of G. Since v is a detour interior vertex of G, there exists a vertex w distinct from u and v such that v lies between u and w.
Let $P : u = v_1, v_2, v_3, ..., v = v_j, v_{j+1}, ..., v_k = w$
be a u - w detour, where $1 < j < k$. However, $v_{j+1} \in N(v)$ and $D(u, v_{j+1}) > D(u, v)$, which contradicts that v is a detour boundary vertex of u.

For the converse, let v be a vertex that is not a detour interior vertex of G. Hence there exists some vertex u such that for every vertex w distinct from u and v, the vertex v does not lie between u and w. Let $x \in N(v)$, Then $D(u, x) \leq D(u, v)$, that is v is a detour boundary vertex of u.

Remark 6.9. Note that based on standard distance cut vertex is not a boundary vertex. But the converse is not true. That is not a boundary vertex need not imply cut vertex.
Consider the graph in Figure 2, with respect to standard distance, vertex e is neither boundary vertex nor a cut vertex.

With respect to detour distance we have the following theorem.

Theorem 6.10. *A vertex v in a connected graph G is detour boundary vertex if and only if v is not a cut vertex.*

Proof. Assume, to the contrary that, there exists a connected graph $G:(V,E)$ and a cut vertex v of G such that v is a detour boundary vertex of some vertex u in G. Let G_1 be component of $G-v$ that contains u and let G_2 be another component of $G-v$. If w is a neighbor of v that belong to G_2 then $D(u,w)>D(u,v)$, which contradicts our assumption that v is a detour boundary vertex of u.

Conversely assume v is not a cut vertex of G. To prove v is detour boundary vertex of G. It is enough to prove v is not a detour interior vertex (Theorem.6.8). Assume on the contrary that v is detour interior vertex of G. That is for every vertex (say) x distinct from v there exists a vertex y (say) such that v lies between x and y. That is $D(x,y)=D(x,v)+D(v,y)$. Then we claim that all $x-y$ paths pass through v. Let P_1 be an $x-y$ path which pass through v. If not there exists another $x-y$ path in G which do not contain v (say P_2). Then $P_1 \bigcup P_2$ contain a cycle in G. Then there exists at least one arc of P_2 which is common to the $x-v$ detour and in the $v-y$ detour. Then the detour path from x to y contains y and the detour path from v to y contains x. Thus $D(x,y)<D(x,v)+D(v,y)$. Hence our assumption is wrong and thus all $x-y$ paths pass through v. Hence v is a cut vertex of G, which is a contradiction. Hence our assumption is wrong. Therefore v is not a detour interior vertex.

Remark 6.11. It follows from the Theorem 6.10 that a connected graph without cut vertices is detour self boundary graph.

Remark 6.12. With respect to standard distance, interior vertex need not be cut vertex. For example in Figure 2, here e is interior vertex but not a cut vertex .

With respect to detour distance we have the following theorem.

Theorem 6.13. *Let $G:(V,E)$ be a connected graph. A vertex v is detour interior vertex if and only if it is a cut vertex of G.*

Proof. Assume v is a cut vertex. Then v is not a detour boundary vertex (Theorem 6.10). Then v is detour interior vertex (Theorem 6.8).

Conversely assume v is detour interior vertex. Then v is not a detour boundary vertex (Theorem 6.8). Hence v is a cut vertex (Theorem 6.10).

Theorem 6.14. *Let $G : (V, E)$ be a connected graph. A vertex v of G is a detour boundary vertex of every vertex distinct from v if and only if v is a complete vertex.*

Proof. First let v be a complete vertex in G, and let u be a vertex distinct from v. Also, let $u{=}v_0, v_1, ..., v_k = v$ be a u-v detour and let w be a neighbor of v.
Then there are two cases.

Case.1
If $w = v_{k-1}$, then $D(u, w) \leq D(u, v)$.
$\Longrightarrow v$ is a detour boundary vertex of u.
Case.2
If $w \neq v_{k-1}$, since v is complete, arc $(w, v_{k-1}) \in E(G)$ and $u{=}v_0, v_1, ...,$ v_{k-1}, w is a u-w path in G.
$\Longrightarrow D(u, w) \leq D(u, v)$.
Hence v is a detour boundary vertex of u.

For the converse, assume v is a detour boundary vertex of every vertex distinct from v. Let v is a boundary vertex of u (say). Let w_1 and w_2 be two neighbors of v. Assume to the contrary that $(w_1, w_2) \notin E(G)$. Since v is a boundary vertex of u, $D(u, v) \geq D(u, w_1)$ and $D(u, v) \geq D(u, w_2)$. Since $(w_1, w_2) \notin E(G)$, $\exists$ detours $u - v - w_1$ and $u - v - w_2$ such that $D(u, v) < D(u, w_1)$ and $D(u, v) < D(u, w_2)$. which is a contradiction to the assumption that v is a boundary vertex of u. Hence $(w_1, w_2) \in E(G)$. Therefore v is complete.

Remark 6.15. In a connected graph every detour eccentric vertex is detour boundary vertex but a detour boundary vertex need not be detour eccentric vertex.

Example 6.16. Consider the graph given in Figure 2, here e is detour boundary vertex but not detour eccentric.

7. Conclusion and Application

In this paper we introduce the concepts of detour boundary vertex and detour interior vertex in a graph. Necessary and sufficient conditions for a vertex to be detour boundary vertex and detour interior vertex are obtained. Also it is proved that a vertex v is detour interior vertex of a graph $G : (V, E)$ if and only if it is a cut vertex of G. Hence using results in this paper we can easily identify detour boundary vertex and detour interior vertex of a graph.

Overlay network provide base infrastructures for many areas including multimedia streaming and content distributions. Since most overlay networks are highly decentralized, cut vertex may exists in such systems due to the lack of centralized management. A cut vertex is defined as a network node whose removal increases the number of network components. Failure of these nodes can break an overlay into a larger number of disconnected components and greatly downgrade the upper layer services like media streaming. Hence identifying cut vertices of the network is required.

For a variety of reasons, hexahedral elements are often preferred over tetrahedral elements for use in finite element analysis. However, unlike tetrahedral meshes, hexahedral meshes are much more constrained and, therefore, much more difficult to generate. Sweeping is a technology that has received significant research in the past few years and is a method of meshing two and one half dimensional volumes with an all hex mesh. One problem with the current sweeping technology is how to determine the location of the new vertices created on the interior of each barrel. Thus the identification of cut vertices and interior vertices through graphs is useful.

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GENERALIZED k-HORADAM HYBRID NUMBERS

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Abstract. The aim of this paper is to define the generalized k-Horadam hybrid numbers and to derive with Binet formula, generating function, exponential generating function, generating matrix, Catalan's identity, Cassini identity and d'Ocagne's identity for the generalized k-Horadam hybrid numbers.

1. Introduction

In the literature, there are various studies involving hybrid numbers and Horadam numbers. The generalized k-Horadam sequence was defined by Yasin Yazlik and Necati Taskara [9]. Mustafa Ozdemir has given information about hybrid numbers [6]. Later, Anetta Szynal [1], [2], Paula Catarino [7] and Esra Erkan [4] extended the concept of hybrid numbers as Jacobsthal, Jacobsthal-Lucas hybrid numbers, k-Pell hybrid numbers, k-Fibonacci and k-Lucas hybrid numbers. The study by Tuncay Deniz Senturk on Horadam hybrid numbers has been a great inspiration for writing this paper. In this study, we have introduced the generalized k-Horadam hybrid numbers. Further, we have discussed few important properties, generating function, Binet formula and Vajda's identity for the generalized k-Horadam hybrid numbers.

2. Basic definitions

Hybrid numbers. A new non-commutative number system referred as the hybrid numbers was defined by Mustafa Ozdemir [6]. Hybrid numbers are the combinations of complex, hyperbolic and dual numbers. It is denoted

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by $\mathcal{K}$.

$\mathcal{K} = \left\{z = a + ib + \epsilon c + hd; a, b, c, d \in R, i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = -hi = \epsilon + i\right\}.$

For further information on the algebraic and geometric features of the hybrid number system, one may refer [6].

The generalized k-Horadam numbers. Let k be any positive real number and $f(k)$, $g(k)$ are the scalar valued polynomials. For $n \geq 0$ and $f^2(k) + 4g(k) > 0$, the generalized k-Horadam sequence has recursive equation

$$H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}, \tag{1}$$

with initial conditions $H_{k,0} = a$ and $H_{k,1} = b$. The above equation is the second-order linear differential equation and its characteristic equation is $\lambda^2 = f(k)\lambda + g(k)$. The two real roots of these equations are

$$r_1 = \frac{f(k) + \sqrt{f^2(k) + 4g(k)}}{2}, r_2 = \frac{f(k) - \sqrt{f^2(k) + 4g(k)}}{2}, \tag{2}$$

where $r_1 > r_2$.

Binet formula. For every $n \in N, H_{k,n} = \dfrac{Xr_1^n - Yr_2^n}{r_1 - r_2}$, where $X = b - ar_2$ and $Y = b - ar_1$.
The above equation can also be written as

$$H_{k,n} = Xr_1^n + Yr_2^n, \tag{3}$$

where $X = \dfrac{b - ar_2}{r_1 - r_2}$ and $Y = \dfrac{ar_1 - b}{r_1 - r_2}$.

Generating function.

$$\sum_{i=0}^{\infty} H_{k,i}x^i = \frac{H_{k,0} + x(H_{k,1} - f(k)H_{k,0})}{1 - f(k)x - g(k)x^2}. \tag{4}$$

3. Main Result

In this section, we will define the generalized k-Horadam hybrid number $\{GH_{k,n}\}$ by the recursive relation

$$GH_{k,n} = H_{k,n} + iH_{k,n+1} + \epsilon H_{k,n+2} + hH_{k,n+3}, \tag{5}$$

with initial conditions

$$GH_{k,0} = H_{k,0} + iH_{k,1} + \epsilon H_{k,2} + hH_{k,3}.$$

$$GH_{k,1} = H_{k,1} + iH_{k,2} + \epsilon H_{k,3} + hH_{k,4}.$$

Example. k-Fibonacci hybrid number can be represented as

$$GF_{k,n} = F_{k,n} + iF_{k,n+1} + \epsilon F_{k,n+2} + hF_{k,n+3},$$

with initial conditions

$$GF_{k,0} = F_{k,0} + iF_{k,1} + \epsilon F_{k,2} + hF_{k,3}.$$
$$GF_{k,1} = F_{k,1} + iF_{k,2} + \epsilon F_{k,3} + hF_{k,4}.$$

Addition and subtraction of the generalized k-Horadam hybrid numbers. Let $GH^1_{k,n}$ and $GH^2_{k,n}$ are any two generalized k-Horadam hybrid numbers, then

$$\begin{aligned} GH^1_{k,n} + GH^2_{k,n} &= \left(H^1_{k,n} + iH^1_{k,n+1} + \epsilon H^1_{k,n+2} + hH^1_{k,n+3}\right) \\ &\quad + \left(H^2_{k,n} + iH^2_{k,n+1} + \epsilon H^2_{k,n+2} + hH^2_{k,n+3}\right), \\ &= (H^1_{k,n} + H^2_{k,n}) + i(H^1_{k,n+1} + H^2_{k,n+1}) \\ &\quad + \epsilon(H^1_{k,n+2} + \epsilon H^2_{k,n+2}) + h(H^1_{k,n+3} + H^2_{k,n+3}). \\ GH^1_{k,n} - GH^2_{k,n} &= \left(H^1_{k,n} + iH^1_{k,n+1} + \epsilon H^1_{k,n+2} + hH^1_{k,n+3}\right) - \\ &\quad \left(H^2_{k,n} + iH^2_{k,n+1} + \epsilon H^2_{k,n+2} + hH^2_{k,n+3}\right), \\ &= (H^1_{k,n} - H^2_{k,n}) + i(H^1_{k,n+1} - H^2_{k,n+1}) \\ &\quad + \epsilon(H^1_{k,n+2} - \epsilon H^2_{k,n+2}) + h(H^1_{k,n+3} - H^2_{k,n+3}). \end{aligned}$$

Multiplication of the generalized k-Horadam hybrid numbers. Let $GH^3_{k,n}$ and $GH^4_{k,n}$ are any two generalized k-Horadam hybrid numbers, then

$$\begin{aligned} GH^3_{k,n}.GH^4_{k,n} &= \left(H^3_{k,n} + iH^3_{k,n+1} + \epsilon H^3_{k,n+2} + hH^3_{k,n+3}\right) \\ &\quad \left(H^4_{k,n} + iH^4_{k,n+1} + \epsilon H^4_{k,n+2} + hH^4_{k,n+3}\right), \\ &= H^3_{k,n}H^4_{k,n} - H^3_{k,n+1}H^4_{k,n+1} + H^3_{k,n+3}H^4_{k,n+3} \\ &\quad + H^3_{k,n+1}H^4_{k,n+2} + H^3_{k,n+2}H^4_{k,n+1} \\ &\quad + i\left(H^3_{k,n}H^4_{k,n+1} + H^3_{k,n+1}H^4_{k,n} + H^3_{k,n+1}H^4_{k,n+3} - H^3_{k,n+3}H^4_{k,n+1}\right) \\ &\quad + \epsilon\left(H^3_{k,n}H^4_{k,n+2} + H^3_{k,n+1}H^4_{k,n+3} + H^3_{k,n+2}H^4_{k,n} - H^3_{k,n+2}H^4_{k,n+3}\right) \\ &\quad + \epsilon\left(-H^3_{k,n+3}H^4_{k,n+1} + H^3_{k,n+3}H^4_{k,n+2}\right) \\ &\quad + h\left(H^3_{k,n}H^4_{k,n+1} - H^3_{k,n+1}H^4_{k,n+2} + H^3_{k,n+2}H^4_{k,n+1} + H^3_{k,n+3}H^4_{k,n}\right). \end{aligned}$$

Scalar and vector parts of the generalized k-Horadam hybrid numbers. Let $GH_{k,n}$ be the n^{th} k-Horadam hybrid number. The scalar and vector parts of these numbers are denoted by

$S_{GH_{k,n}} = H_{k,n}$, $V_{GH_{k,n}} = iH_{k,n+1} + \epsilon H_{k,n+2} + hH_{k,n+3}$.
Hence, $GH_{k,n} = S_{GH_{k,n}} + V_{GH_{k,n}}$

Conjugate of the generalized k-Horadam hybrid numbers.
Let $GH_{k,n} = H_{k,n} + iH_{k,n+1} + \epsilon H_{k,n+2} + hH_{k,n+3}$ be a n^{th} k-Horadam hybrid number. Its conjugate is defined by
$GH_{k,n} = H_{k,n} - iH_{k,n+1} - \epsilon H_{k,n+2} - hH_{k,n+3}$.

Character of the generalized k-Horadam hybrid numbers.

$$\begin{aligned} C(GH_{k,n}) = (H_{k,n})^2(1 - f^2(k)g^2(k)) \\ + H_{k,n}H_{k,n+1}\left(-2g(k) - 2f^3(k)g(k) - 2f(k)g^2(k)\right) \\ + (H_{k,n+2})^2\left(1 - 2f(k) - f^4(k) - 2f^2(k)g(k) - g^2(k)\right). \end{aligned}$$

Norm of the generalized k-Horadam hybrid numbers.

$$N(GH_{k,n}) = \sqrt{|C(GH_{k,n})|}.$$

It can be simplified as
$N(GH_{k,n}) = \sqrt{|(H_{k,n})^2 + (H_{k,n+1} - H_{k,n+2})^2 - (H_{k,n+2})^2 - (H_{k,n+3})^2|}$.

Matrix representation of the generalized k-Horadam hybrid numbers. Using the approach of Mustafa Ozdemir [6] and Tuncay Deniz Senturk [8], we can define the following matrix which is also used to compute the character and norm of the generalized k-Horadam hybrid numbers.

$$G\mathcal{H}_{k,n} = \begin{bmatrix} H_{k,n} + H_{k,n+2} & H_{k,n+1} - H_{k,n+2} + H_{k,n+3} \\ H_{k,n+2} - H_{k,n+1} + H_{k,n+3} & H_{k,n} - H_{k,n+2} \end{bmatrix}$$

The definition of the generalized k-Horadam hybrid number is reduced to certain special cases which is tabulated below.

Table 1

S.no.	Name of the sequences	$(a, b, f(k), g(k))$
1	Generalized k-Fibonacci sequence	$(a, b, k, 1)$
2	k-Fibonacci sequence	$(0, 1, k, 1)$
3	k-Lucas sequence	$(2, k, k, 1)$
4	Horadam sequence	(a, b, p, q)
5	Fibonacci sequence	$(0, 1, 1, 1)$
6	Lucas sequence	$(2, 1, 1, 1)$
7	Pell sequence	$(0, 1, 2, 1)$
8	Jacobsthal sequence	$(0, 1, 1, 2)$
9	Jacobsthal-Lucas sequence	$(2, 1, 1, 2)$
10	Mersenne sequence	$(0, 1, 3, -2)$
11	Fermat sequence	$(1, 3, 3, 2)$

Binet formula for the generalized k-Horadam hybrid numbers.

Theorem 1. Let $n \geq 0$ be an integer, then

$$GH_{k,n} = Xr_1^n(1 + ir_1 + \epsilon r_1^2 + hr_1^3) + Yr_2^n(1 + ir_2 + \epsilon r_2^2 + hr_2^3), \quad (6)$$

where $X = \dfrac{b - ar_2}{r_1 - r_2}$ and $Y = \dfrac{ar_1 - b}{r_1 - r_2}$.

Proof. Using (3) and (5),

$$\begin{aligned} GH_{k,n} &= H_{k,n} + iH_{k,n+1} + \epsilon H_{k,n+2} + hH_{k,n+3}, \\ &= (Xr_1^n + Yr_2^n) + i(Xr_1^{n+1} + Yr_2^{n+1}) \\ &\quad + \epsilon(Xr_1^{n+2} + Yr_2^{n+2}) + h(Xr_1^{n+3} + Yr_2^{n+3}), \\ &= Xr_1^n \left(1 + ir_1 + \epsilon r_1^2 + hr_1^3\right) + Yr_2^n \left(1 + ir_2 + \epsilon r_2^2 + hr_2^3\right). \end{aligned}$$

Generating function of the generalized k-Horadam hybrid numbers.

Theorem 2.

$$\mathcal{G}_{GH_{k,n}(t)} = \sum_{i=0}^{\infty} GH_{k,i}t^i = \frac{(GH_{k,0} + t(GH_{k,1} - f(k)GH_{k,0}))}{1 - f(k)t - g(k)t^2}. \quad (7)$$

Proof. Let $\mathcal{G}_{GH_{k,n}}$ be the generating function of the generalized k-Horadam hybrid number, then

$$\mathcal{G}_{GH_{k,n}(t)} = \sum_{i=0}^{\infty} GH_{k,i}t^i = GH_{k,0} + tGH_{k,1} + t^2GH_{k,2} + ... + t^nGH_{k,n} + ... \quad (8)$$

Multiplying (8) with $f(k)t$ and $g(k)t^2$, respectively, we have

$$f(k)t\mathcal{G}_{GH_{k,n}(t)} = f(k)tGH_{k,0} + f(k)t^2GH_{k,1} + ... + f(k)t^{n+1}GH_{k,n} + ... \quad (9)$$

$$g(k)t^2\mathcal{G}_{GH_{k,n}(t)} = g(k)t^2GH_{k,0} + g(k)t^3GH_{k,1} + ... + g(k)t^{n+2}GH_{k,n} + ... \quad (10)$$

From (8), (9) and (10) we get

$$\begin{aligned} (1 - f(k)t - g(k)t^2)\mathcal{G}_{GH_{k,n}(t)} &= GH_{k,0} + t(GH_{k,1} - f(k)GH_{k,0}), \\ \mathcal{G}_{GH_{k,n}(t)} &= \frac{GH_{k,0} + t(GH_{k,1} - f(k)GH_{k,0})}{1 - f(k)t - g(k)t^2}. \end{aligned}$$

Summation formula for the generalized k-Horadam hybrid numbers.

Theorem 3. For every positive integer n,

$$\sum_{t=1}^{n} GH_{k,t} = \frac{1}{f(k) + g(k) - 1}\left(GH_{k,n+1} + g(k)GH_{k,n} - GH_{k,1} + g(k)GH_{k,0}\right). \quad (11)$$

Proof. We can prove this result by mathematical induction.
Let $n = 1$

$$GH_{k,1} = \frac{1}{f(k)+g(k)-1}\left(GH_{k,2} + g(k)GH_{k,1} - GH_{k,1} + g(k)GH_{k,0}\right), \tag{12}$$

We know that
$GH_{k,0} = a$, $GH_{k,1} = b$, $GH_{k,2} = f(k)b + g(k)a$.
Equation (12) can be written as

$$b = \frac{1}{f(k)+g(k)-1} b(f(k)+g(k)-1),$$

On simplification, we can say the result is true for $n = 1$.
Assume that the result is true for $n = m$.

$$\sum_{t=1}^{m} GH_{k,t} = \frac{1}{f(k)+g(k)-1}\left(GH_{k,m+1} + g(k)GH_{k,m} - GH_{k,1} - g(k)GH_{k,0}\right).$$

To prove the result is true for $n = m + 1$

$$\sum_{t=1}^{m+1} GH_{k,t} = \frac{1}{f(k)+g(k)-1}\left(GH_{k,m+2} + g(k)GH_{k,m+1} - GH_{k,1} - g(k)GH_{k,0}\right),$$

$$\sum_{t=1}^{m+1} GH_{k,t} = GH_{k,m+1} + \frac{1}{f(k)+g(k)-1}\left(GH_{k,m+1} + g(k)GH_{k,m} - GH_{k,1} - g(k)GH_{k,0}\right),$$

$$= \frac{f(k)GH_{k,m+1} + g(k)GH_{k,m} + g(k)GH_{k,m+1} - GH_{k,1} - g(k)GH_{k,0}}{f(k)+g(k)-1},$$

Using simple mathematical simplifications, we can get the result.

Relationship between $\bar{r_1}$ and $\bar{r_2}$

Let us consider,

$$\bar{r_1} = 1 + ir_1 + \epsilon r_1^2 + hr_1^3, \quad \bar{r_2} = 1 + ir_2 + \epsilon r_2^2 + hr_2^3,$$

From (2),

$$r_1 + r_2 = f(k), \quad r_1 - r_2 = \sqrt{f^2(k) + 4g(k)},$$

$$\text{Let } \Delta = \sqrt{f^2(k) + 4g(k)}, \quad \text{then we can write } \ r_1 - r_2 = \Delta,$$

$$r_1^2 = r_1 f(k) + g(k), \quad r_2^2 = r_2 f(k) + g(k),$$

$$\text{Let } \theta = 1 - g(k) + f(k)g(k) + g(k)^3,$$

$$\eta = -iu_2 + \epsilon(-g(k))u_1 - u_2 + hu_1,$$

$$U_0 = iu_1 + \epsilon u_2 + hu_3,$$

$$\text{Now } \bar{r_1}\bar{r_2} = (1 + ir_1 + \epsilon r_1^2 + hr_1^3)(1 + ir_2 + \epsilon r_2^2 + hr_2^3),$$

After some simplification, we get
$\bar{r_1}\bar{r_2} = 2\bar{r_2} - \theta + \Delta(U_0 - g(k)\eta)$.
Similarly, we can try it for the remaining values. The results are displayed in Table 2.

Table 2

	$\bar{r_1}$	$\bar{r_2}$
$\bar{r_1}$	$2\bar{r_1} - c(\bar{r_1})$	$2\bar{r_2} - \theta + \Delta(U_0 - g(k)\eta)$
$\bar{r_2}$	$2\bar{r_1} - \theta - \Delta(U_0 - g(k)\eta)$	$2\bar{r_2} - c(\bar{r_2})$

Vajda's identity.

Theorem 4. For any integers n, r and s, we have

$$GH_{k,n+r}GH_{k,n+s} - GH_{k,n}GH_{k,n+r+s} = XY(-g(k))^n\Delta^2 u_r \times \left(-2U_s + \theta u_s + v_s(U_0 - g(k)\eta)\right), \tag{13}$$

where

$$X = \frac{b - ar_2}{r_1 - r_2}, \quad Y = \frac{ar_1 - b}{r_1 - r_2}, \quad r_1 > r_2,$$
$$U_s = u_s + iu_{s+1} + \epsilon u_{s+2} + hu_{s+3}, \quad U_0 = iu_1 + \epsilon u_2 + hu_3,$$
$$u_s = \frac{r_1^s - r_2^s}{r_1 - r_2} \text{ (Binet formula for Fibonacci number)},$$
$$v_s = r_1^n + r_2^n \text{ (Binet formula for Lucas number)},$$
$$\theta = 1 - g(k) + f(k)g(k) + g(k)^3,$$
$$\Delta = r_1 - r_2, \quad u_r = \frac{r_1^r - r_2^r}{\Delta}, \quad \eta = -iu_2 + \epsilon\left(-g(k)u_1 - u_2\right) + hu_1.$$

Proof. From (6),

$$GH_{k,n} = Xr_1^n\bar{r_1} + Yr_2^n\bar{r_2},$$
$$GH_{k,n+r}GH_{k,n+s} - GH_{k,n}GH_{k,n+r+s}$$
$$= (X\bar{r_1}r_1^{n+r} + Y\bar{r_2}r_2^{n+r})(X\bar{r_1}r_1^{n+s} + Y\bar{r_2}r_2^{n+s})$$
$$- (X\bar{r_1}r_1^n + Y\bar{r_2}r_2^n)(X\bar{r_1}r_1^{n+r+s} + Y\bar{r_2}r_2^{n+r+s}),$$
$$= XY\bar{r_1}\bar{r_2}r_1^{n+r}r_2^{n+s} + XY\bar{r_2}\bar{r_1}r_2^{n+r}r_1^{n+s}$$
$$- XY\bar{r_1}\bar{r_2}r_1^n r_2^{n+r+s} - YX\bar{r_2}\bar{r_1}r_2^n r_1^{n+r+s},$$
$$= XY\left(\bar{r_1}\bar{r_2}r_1^{n+r}r_2^{n+s} + \bar{r_2}\bar{r_1}r_2^{n+r}r_1^{n+s} - \bar{r_1}\bar{r_2}r_1^n r_2^{n+r+s} - \bar{r_2}\bar{r_1}r_2^n r_1^{n+r+s}\right),$$
$$= XY(r_1r_2)^n\left(\bar{r_1}\bar{r_2}r_2^s - \bar{r_2}\bar{r_1}r_1^s\right)\left(r_1^r - r_2^r\right),$$

$$
\begin{aligned}
&= XY(-g(k))^n \Delta u_r \left((2\bar{r_2} - \theta + \Delta(U_0 - g(k)\eta))\, r_2^s \right) \\
&\quad - XY(-g(k))^n \Delta u_r \left((2\bar{r_1} - \theta - \Delta(U_0 - g(k)\eta))\, r_1^s \right), \\
&= XY(-g(k)^n \Delta u_r(-2(\bar{r_1} r_1^s - \bar{r_2} r_2^s) + \theta(r_1^s - r_2^s))) \\
&\quad + XY(-g(k)^n \Delta u_r(\Delta(U_0 - g(k)\eta)(r_1^s + r_2^s)), \\
&= XY(-g(k))^n \Delta^2 u_r(-2U_s + \theta u_s + v_s(U_0 - g(k)\eta)).
\end{aligned}
$$

Particular cases.

Catalan's identity. For $r = -s$ in (13), we get

$$
\begin{aligned}
GH_{k,n-s}GH_{k,n+s} - GH_{k,n}GH_{k,n} = -XY(-g(k))^{n-s}\Delta^2 u_s \\
\left(-2U_s + \theta u_s + v_s(U_0 - g(k)\eta)\right).
\end{aligned}
$$

Cassini identity. For $s = 1, r = -1$ in (13), we get

$$
\begin{aligned}
GH_{k,n-1}GH_{k,n+1} - (GH_{k,n})^2 = -XY(-g(k))^{n-1}\Delta^2 \\
\left(-2U_1 + \theta + f(k)(U_0 - g(k)\eta)\right).
\end{aligned}
$$

d'Ocagne's identity. For $s = m - n, r = -1$ in (13), we get

$$
\begin{aligned}
GH_{k,n+1}GH_{k,n} - GH_{k,n}GH_{k,m+1} = XY(-g(k))^n \\
\left(-2U_{m-n} + \theta u_{m-n} + v_{m-n}(U_0 - g(k)\eta)\right).
\end{aligned}
$$

Example. Let $(a, b, f(k), g(k)) = (0, 1, 1, 1)$
when $k = 1$, the sequence $\{GH_{k,n}\}_{n=0}^{\infty}$ reduces to Fibonacci hybrid sequence $\{GF_{1,n}\}_{n=0}^{\infty}$. Using Theorem 4, we can get
$r_1 = \dfrac{1+\sqrt{5}}{2}, r_2 = \dfrac{1-\sqrt{5}}{2}$, $X = \dfrac{1}{\sqrt{5}}, Y = \dfrac{-1}{\sqrt{5}}$,
$g(k) = 1, \Delta^2 = 5$, $u_s = F_s, v_s = L_s$,
$U_s = F_s + iF_{s+1} + \epsilon F_{s+2} + hF_{s+3}, U_0 = i + \epsilon + 2h$, $\theta = 2, \eta = -i - 2\epsilon + h$.
Here $\{U_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ gives $\{F_n\}_{n=0}^{\infty}$ and $\{L_n\}_{n=0}^{\infty}$, when F_n and L_n are the n^{th} Fibonacci and Lucas numbers. We can obtain Vajda's, Catalan's and Cassini identities for Fibonacci hybrid number.

Vajda's identity.

$$
\begin{aligned}
&GF_{1,n+r}GF_{1,n+s} - GF_{1,n}GF_{1,n+r+s} \\
&\quad = (-1)^n F_r \left(2(iF_{s+1} + \epsilon F_{s+2} + hF_{s+3}) - L_s(2i + 3\epsilon + h)\right).
\end{aligned}
$$

Catalan's identity.

$$
\begin{aligned}
&GF_{1,n-s}GF_{1,n+s} - GF_{1,n}^2 \\
&\quad = (-1)^{n+s+1} F_s \left(2(iF_{s+1} + \epsilon F_{s+2} + hF_{s+3}) - L_s(2i + 3\epsilon + h)\right).
\end{aligned}
$$

Cassini identity.

$$GF_{1,n-1}GF_{1,n+1} - GF_{1,n}^2 = (-1)^n (\epsilon + 5h).$$

Exponential generating function for the generalized k-Horadam hybrid numbers.
Theorem 5.

$$G_e(t) = X(1 + ir_1 + \epsilon r_1^2 + hr_1^3)e^{r_1 t} + Y(1 + ir_2 + \epsilon r_2^2 + hr_2^3)e^{r_2 t}. \tag{14}$$

Proof. To do this, we first define the following series

$$G_e(t) = \sum_{n=0}^{\infty} GH_{k,n}\frac{t^n}{n!}$$

Considering the Maclaurin expansion for the exponential function and Equation (6)

$$\begin{aligned} G_e(t) &= \sum_{n=0}^{\infty} \left(Xr_1^n(1 + ir_1 + \epsilon r_1^2 + hr_1^3) + Yr_2^n(1 + ir_2 + \epsilon r_2^2 + hr_2^3)\right) \frac{t^n}{n!}, \\ &= X(1 + ir_1 + \epsilon r_1^2 + hr_1^3)\frac{(r_1 t)^n}{n!} + Y(1 + ir_2 + \epsilon r_2^2 + hr_2^3)\frac{(r_2 t)^n}{n!}, \\ &= X(1 + ir_1 + \epsilon r_1^2 + hr_1^3)e^{r_1 t} + Y(1 + ir_2 + \epsilon r_2^2 + hr_2^3)e^{r_2 t}. \end{aligned}$$

4. Conclusion

The current study aimed to present a focus on the generalized k-Horadam hybrid number process. However, it was studied in the light of Mustafa Ozdemir's relation. The k-Horadam hybrid numbers are the combination of complex, hyperbolic and dual numbers. In this study, we introduced Binet formula, generating function, matrix representation, Catalan's identity, Cassini identity and d'Ocagne's identity. In future studies, it is also possible to highlight the generalized k-Horadam hybrid numbers in polynomials, transformations and factorials.

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CONTINUED FRACTIONS FOR BILATERAL BASIC HYPERGEOMETRIC SERIES

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Abstract. In this paper, using the asymmetric bilateral Bailey transform, certain transformations along with continued fractions of ${}_4\psi_5$, ${}_3\psi_4$ and ${}_2\psi_3$ bilateral basic hypergeometric series have been established.

1. Introduction

S. P. Singh [7] has discovered very nice transformation formulae that convert basic bilateral hypergeometric functions to basic hypergeometric functions. These formulae are used to generate our key findings in this paper. Part 1 of this study focuses on utilizing Bailey's transformation to derive the summation formula for ${}_2\phi_1$ basic hypergeometric series. In the following section, the summation formulae for ${}_4\psi_5$ and ${}_3\psi_4$ are determined using an asymmetric bilateral Bailey's transformation, followed by continued fractions for the same. The last section concentrates on the continued fraction of ${}_2\psi_3$. We have also deduced some interesting special cases.

2. Definitions and notations

For any number a and q real or complex such that $|q| < 1$, we have:

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

$$(a_1,a_2,a_3,...,a_r;q)_n = (a_1;q)_n(a_2;q)_n(a_3;q)_n...(a_r;q)_n.$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{2.1}$$

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$$(a;q)_{n-r} = \frac{(a;q)_n(-q/a)^r q^{\frac{r(r-1)}{2}-nr}}{(\frac{q^{1-n}}{a};q)_r}. \tag{2.2}$$

$$\lim_{q\to 1}\frac{(q^a;q)_n}{(1-q)^n} = (a)_n. \tag{2.3}$$

A ratio of the following type is called a continued fraction:

$$a_0 + \frac{a_1}{a_2 +}\frac{a_3}{a_4 +}\frac{a_5}{a_6 + ...},$$

where $a_0, a_1, a_2, a_3, ...$ are real or complex numbers.

A generalized basic hypergeometric series is given by:

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, a_3, ...a_r; q; z \\ b_1, b_2, b_3, ..., b_s \end{matrix}\right) = \sum_{n=0}^{\infty}\frac{(a_1, a_2, .., a_r; q)_n z^n \left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r}}{(b_1, b_2, ..., b_s; q)_n (q;q)_n},$$

where $|z| < 1$, $|q| < 1$.

A generalized basic bilateral hypergeometric series is defined as:

$${}_r\psi_s\left(\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix}\right) = \sum_{n=-\infty}^{\infty}\frac{(a_1, a_2, .., a_r; q)_n z^n}{(b_1, b_2, ..., b_s; q)_n}\left\{(-1)^n q^{n(n-1)/2}\right\}^{s-r},$$

where $|z| < 1$, $|q| < 1$ and $r < s + 1$.

A bilateral hypergeometric series is defined as:

$${}_pH_p\left(\begin{matrix} a_1, a_2, .., a_p; q; z \\ b_1, b_2, ..., b_p \end{matrix}\right) = \sum_{n=-\infty}^{\infty}\frac{(a_1)_n(a_2)_n..(a_p)_n z^n}{(b_1)_n(b_2)_n...(b_p)_n}.$$

A truncated basic hypergeometric series is defined as:

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, a_3, ..., a_r; q; z \\ b_1, b_2, b_3, ..., b_s \end{matrix}\right)_N = \sum_{n=0}^{N}\frac{(a_1, a_2, .., a_r; q)_n z^n}{(b_1, b_2, .., b_s; q)_n (q;q)_n},$$

where $|z| < 1$, $|q| < 1$ and no zero appears in the denominator.

W. N. Bailey established the Bailey transform in 1947. It states that if

$$\beta_n = \sum_{r=0}^{n}\alpha_r u_{n-r} v_{n+r}, \tag{2.4}$$

$$\gamma_n = \sum_{r=0}^{\infty}\delta_{r+n} u_r v_{r+2n}, \tag{2.5}$$

then, subject to the convergence conditions,

$$\sum_{n=0}^{\infty}\alpha_n\gamma_n = \sum_{n=0}^{\infty}\beta_n\delta_n. \tag{2.6}$$

Andrews [2], extended the Bailey transform to include *Asymmetric Bilateral Bailey Transforms.* It is as follows:
Let $m = max\{n, -(n+1)\}$. If

$$\beta_n = \sum_{r=-n-1}^{n} \alpha_r u_{n-r} v_{n+r+1}, \tag{2.7}$$

$$\gamma_n = \sum_{r=m}^{\infty} \delta_r u_{r-n} v_{r+n+1}, \tag{2.8}$$

then, subject to the convergence conditions,

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{2.9}$$

3. Preliminary Results

We are going to use following summation formulae to establish the main results.

From Agarwal [1, App II, (8)], we have

$$ {}_2\phi_1 \left(\begin{matrix} a, y; q; q \\ ayq \end{matrix} \right)_n = \frac{(aq, yq; q)_n}{(q, ayq; q)_n}. \tag{3.1}$$

In Gasper and Rahman [4, eq(1.5.1), p. 14], we found that

$$ {}_2\phi_1 \left(\begin{matrix} a, b; q; \frac{c}{ab} \\ c \end{matrix} \right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \qquad |c/ab| < 1. \tag{3.2}$$

From Slater [8, eq(6.1.2.3), p. 182], we have

$$ {}_AH_A\left[(a);(b);z\right] = {}_{A+1}F_A\left[1,(a);(b);z\right] + \prod_{v=1}^{A} \frac{(1-b_v)}{(1-a_v)} {}_{A+1}F_A\left[1,(2-b);(2-a);1/z\right], \tag{3.3}$$

where $(a) = (a_1, a_2, ..., a_A)$ and $(b) = (b_1, b_2, ..., b_A)$ are real or complex parameters.

According to PETR [5, p. 10], an inverse tangent integral can be written in terms of hypergeometric functions as follows:

$$\int \frac{\arctan x}{x} dx = x\, {}_3F_2 \left(\begin{matrix} 1, \frac{1}{2}, \frac{1}{2}; -x^2 \\ \frac{3}{2}, \frac{3}{2} \end{matrix} \right). \tag{3.4}$$

We discovered in Gasper and Rahman [4, ex.(5.18)(ii), p. 150] that

$$
{}_3\psi_3\left(\begin{array}{c} b,c,d;q;\frac{q^2}{bcd} \\ \frac{q^2}{b},\frac{q^2}{c},\frac{q^2}{d} \end{array}\right)=\frac{(q,\frac{q^2}{bc},\frac{q^2}{bd},\frac{q^2}{cd};q)_\infty}{(\frac{q^2}{b},\frac{q^2}{c},\frac{q^2}{d},\frac{q^2}{bcd};q)_\infty}. \tag{3.5}
$$

Singh [7, eq(3.1), p. 93], discovered that

$$
{}_2\psi_3\left(\begin{array}{c} \alpha,\beta;q;\frac{zq}{\alpha\beta} \\ q/\alpha,q/\beta,0 \end{array}\right)=\frac{(q,q/\alpha\beta;q)_\infty}{(q/\alpha,q/\beta;q)_\infty}{}_4\phi_3\left(\begin{array}{c} \alpha,\beta,z,q/z;q;\frac{q}{\alpha\beta} \\ -q,\sqrt{q},-\sqrt{q} \end{array}\right), \tag{3.6}
$$

where $|q/\alpha\beta|<1$.

Singh [7, eq(3.2), p. 93], established the following transformation formula:

$$
{}_4\psi_5\left(\begin{array}{c} \alpha,\beta,c,d;q;\frac{q^2}{\alpha\beta cd} \\ q/\alpha,q/\beta,q/c,q/d,0 \end{array}\right)=\frac{(q,q/\alpha\beta;q)_\infty}{(q/\alpha,q/\beta;q)_\infty}{}_3\phi_2\left(\begin{array}{c} \alpha,\beta,q/cd;q;q/\alpha\beta \\ q/c,q/d \end{array}\right), \tag{3.7}
$$

where $|q/\alpha\beta|<1$.

Singh [7, eq(3.3), p. 94], also established that

$$
{}_3\psi_3\left(\begin{array}{c} \alpha,\beta,c;q;\frac{q^2}{\alpha\beta c} \\ q/\alpha,q/\beta,q/c \end{array}\right)=\frac{(q,q/\alpha\beta;q)_\infty}{(q/\alpha,q/\beta;q)_\infty}{}_3\phi_2\left(\begin{array}{c} \alpha,\beta,-q/c;q;q/\alpha\beta \\ -q,q/c \end{array}\right), \tag{3.8}
$$

where $|q/\alpha\beta|<1, |q^3/\alpha^2\beta^2c^2|<|q^2/\alpha\beta c|<1$.

Singh [6] has given a continued fraction for ${}_3\phi_2$ as follows:

$$
\frac{{}_3\phi_2\left(\begin{array}{c} a,b,c;q;de/abc \\ d,e \end{array}\right)}{{}_3\phi_2\left(\begin{array}{c} aq,b,c;q;de/abc \\ dq,e \end{array}\right)}=1-\frac{A_0}{\frac{(1-\frac{e}{aq})}{(1-e)}+}\frac{B_0}{1}-\frac{A_1}{\frac{(1-\frac{e}{aq})}{(1-eq)}+}\frac{B_1}{1}-\frac{A_2}{\frac{(1-\frac{e}{aq})}{(1-eq^2)}+}\frac{B_2}{1}-\cdots, \tag{3.9}
$$

for $n=0,1,2,3,...$

$$
A_n=\left(\frac{deq^n}{abc}\right)\frac{(a-dq^n)(1-bq^n)(1-cq^n)}{(1-eq^n)(1-dq^{2n})(1-dq^{2n+1})},
$$

$$
B_n=\left(\frac{e}{aq}\right)\frac{(1-aq^{n+1})(1-\frac{dq^{n+1}}{b})(1-\frac{dq^{n+1}}{c})}{(1-eq^n)(1-dq^{2n+1})(1-dq^{2n+2})}.
$$

4. Main Results and their Proofs

Theorem 4.1. *If a, b, c and q are real or complex parameters such that $|q| < 1$ and $|c/abq| < 1$, then*

$$
{}_2\phi_1\left(\begin{matrix} a, b; q; \frac{c}{abq} \\ c \end{matrix}\right) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}\left[\frac{(1-c/aq)(1-c/bq)}{(1-c/abq)} + \frac{c}{q}\right].
$$

Proof. Choose

$$
\alpha_r = \frac{(a,b;q)_r q^r}{(q;q)_r (abq;q)_r},\ u_r = v_r = 1,\ \delta_r = \frac{(abq;q)_r (\frac{c}{abq^2})^r}{(c;q)_r}.
$$

Then from equations (2.4) and (3.1), we found that

$$
\beta_n = \frac{(aq, bq; q)_n}{(q, abq; q)_n}, \qquad |q| < 1. \tag{4.1}
$$

by using equations (2.5), (3.2) and (2.1), γ_n can be calculated as follows:

$$
\gamma_n = \frac{(abq;q)_n (\frac{c}{abq^2})^n}{(c;q)_{n-1}(1-\frac{c}{abq^2})}, \qquad |c/abq| < 1. \tag{4.2}
$$

from equation (4.2) and α_r, we get that

$$
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \frac{1}{(1-\frac{c}{abq^2})}\left[{}_2\phi_1\left(\begin{matrix} a, b; q; \frac{c}{abq} \\ c \end{matrix}\right) - \frac{c}{q}\frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}\right]. \tag{4.3}
$$

from (4.1) and (3.2), we get

$$
\sum_{n=0}^{\infty} \beta_n \delta_n = \frac{(c/aq, c/bq; q)_\infty}{(c, c/abq^2)_\infty}. \tag{4.4}
$$

Then the theorem follows by using the equations (4.3), (4.4) and (2.6). ☐

Theorem 4.2. *Let α, β, c, d and q be real or complex parameters satisfying the condition $|q^2/\alpha\beta cd| < 1$ then*

$$
\frac{{}_4\psi_5\left(\begin{matrix} \alpha, \beta, c, d; q; \frac{q^2}{\alpha\beta cd} \\ q/\alpha, q/\beta, q/c, q/d, 0 \end{matrix}\right)}{{}_4\psi_5\left(\begin{matrix} \alpha q, \beta, c, dq; q; \frac{q^2}{\alpha\beta cd} \\ q/\alpha, q^2/\beta, q^2/c, q/d, 0 \end{matrix}\right)} = \frac{1}{(1-q/\beta)}
$$

$$
\left[1 - \frac{A_0}{\frac{(1-\frac{1}{\alpha d})}{(1-\frac{q}{d})} +} \frac{B_0}{1} - \frac{A_1}{\frac{(1-\frac{1}{\alpha d})}{(1-\frac{q^2}{d})} +} \frac{B_1}{1} - \frac{A_2}{\frac{(1-\frac{1}{\alpha d})}{(1-\frac{q^3}{d})} +} \frac{B_2}{1} - \cdots\right],
$$

$$A_n = \left(\frac{q^{n+1}}{\alpha\beta}\right) \frac{(\alpha - q^{n+1}/c)(1-\beta q^n)(1-q^{n+1}/cd)}{(1-q^{n+1}/d)(1-q^{2n+1}/c)(1-q^{2n+2}c)},$$
$$B_n = \left(\frac{1}{\alpha q}\right) \frac{(1-\alpha q^{n+1})(1-q^{n+2}/\beta c)(1-dq^{n+1})}{(1-q^{n+1}/d)(1-q^{2n+2}/c)(1-q^{2n+3}/c)}, \quad n = 0, 1, 2, 3, ...$$

Proof. To determine the desired continued fraction, we first find the summation formula for

$${}_4\psi_5 \left(\begin{array}{c} \alpha q, \beta, c, dq; q; \frac{q^2}{\alpha\beta cd} \\ q/\alpha, q^2/\beta, q^2/c, q/d, 0 \end{array} \right), \qquad |q^2/\alpha\beta cd| < 1.$$

Choose

$$\alpha_r = \frac{(-1)^r (c, dq; q)_r q^{\frac{r(r-1)}{2}} (q/cd)^r}{(q^2/c, q/d; q)_r}, u_r = v_r = \frac{1}{(q;q)_r}, \delta_r = (\alpha q, \beta; q)_r (q/\alpha\beta)^r.$$

from equations (2.7) and (2.2), we get

$$\beta_n = \frac{1}{(q;q)_n (q;q)_{n+1}} \sum_{r=-n-1}^{n} \frac{(c, dq, q^{-n}; q)_r (q^{1+n}/cd)^r}{(q^2/c, q/d, q^{2+n}; q)_r}. \tag{4.5}$$

now by substituting $d = dq$ and then $b = q^{-n}$ in (3.5), equation (4.5) becomes

$$\beta_n = \frac{(q/cd;q)_n}{(q, q^2/c, q/d; q)_n}, \qquad |q/cd| < 1. \tag{4.6}$$

If we choose $m = n$ and then put $r = r + n$ in (2.8) then γ_n converted into a simpler form as follows:

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n+1}. \tag{4.7}$$

use theorem (4.1) and then equation (2.1) for (4.7), we derived to

$$\gamma_n = \frac{(q/\alpha, q^2/\beta; q)_\infty}{(q, q/\alpha\beta; q)_\infty} \frac{(\alpha q, \beta; q)_n (q/\alpha\beta)^n}{(q/\alpha, q^2/\beta; q)_n}, \qquad |q/\alpha\beta| < 1. \tag{4.8}$$

with the help of equations (2.9), (4.6) and (4.8), we get

$${}_4\psi_5 \left(\begin{array}{c} \alpha q, \beta, c, dq; q; \frac{q^2}{\alpha\beta cd} \\ q/\alpha, q^2/\beta, q^2/c, q/d, 0 \end{array} \right) = \frac{(q, q/\alpha\beta; q)_\infty}{(q/\alpha; q^2/\beta; q)_\infty} {}_3\phi_2 \left(\begin{array}{c} \alpha q, \beta, q/cd; q; \frac{q}{\alpha\beta} \\ q^2/c, q/d \end{array} \right), \tag{4.9}$$

where $|q^2/\alpha\beta cd| < 1$.

We complete the proof with equations (3.7), (4.9) and the continued fraction (3.9). □

Theorem 4.3. *If α, β, c and q are real or complex parameters such that $|q|<1$ and $|q^2/\alpha\beta c|<1$, then*

$$\frac{{}_3\psi_3\left(\begin{array}{c}\alpha,\beta,c;q;\frac{q^2}{\alpha\beta c}\\ q/\alpha,q/\beta,q/c\end{array}\right)}{{}_3\psi_4\left(\begin{array}{c}\alpha q,\beta,c;q;\frac{q^2}{\alpha\beta c}\\ q/\alpha,q^2/\beta,q^2/c,0\end{array}\right)}=\frac{1}{(1-q/\beta)}$$

$$\left[1-\frac{A_0}{\frac{(1+\frac{1}{\alpha})}{(1+q)}+}\frac{B_0}{1}-\frac{A_1}{\frac{(1+\frac{1}{\alpha})}{(1+q^2)}+}\frac{B_1}{1}-\frac{A_2}{\frac{(1+\frac{1}{\alpha})}{(1+q^3)}+}\frac{B_2}{1}-\cdots\right],$$

$$A_n=\left(\frac{q^{n+1}}{\alpha\beta}\right)\frac{(\alpha-q^{n+1}/c)(1-\beta q^n)(1+q^{n+1}/c)}{(1+q^{n+1})(1-q^{2n+1}/c)(1-q^{2n+2}/c)},$$
$$B_n=\left(\frac{-1}{\alpha}\right)\frac{(1-\alpha q^{n+1})(1-q^{n+2}/\beta c)}{(1-q^{2n+2}/c)(1-q^{2n+3}/c)},\quad n=0,1,2,3,...$$

Proof. To determine the desired continued fraction, we first discover the summation formula for the

$${}_3\psi_4\left(\begin{array}{c}\alpha q,\beta,c;q;\frac{q^2}{\alpha\beta c}\\ q/\alpha,q^2/\beta,q^2/c,0\end{array}\right),\qquad |q^2/\alpha\beta c|<1.$$

Choose

$$\alpha_r=\frac{(c;q)_r q^{\frac{r(r-1)}{2}}(q/c)^r}{(q^2/c;q)_r},\ u_r=v_r=\frac{1}{(q;q)_r},\delta_r=(\alpha q,\beta;q)_r(q/\alpha\beta)^r.$$

from equations (2.7) and (2.2), we get

$$\beta_n=\frac{1}{(q;q)_n(q;q)_{n+1}}\sum_{r=-n-1}^{n}\frac{(c,q^{-n};q)_r(-q^{1+n}/c)^r}{(q^2/c,q^{2+n};q)_r}.\tag{4.10}$$

now substitute $d=q$ and then $b=q^{-n}$ in (3.5), equation (4.10) becomes

$$\beta_n=\frac{(-q/c;q)_n}{(q,-q,q^2/c;q)_n},\qquad |q/c|<1.\tag{4.11}$$

from the values of δ_r, u_r, v_r, we get γ_n same as in equation (4.8). With the help of (2.9), (4.11) and (4.8), we get the following transformation formula:

$${}_3\psi_4\left(\begin{array}{c}\alpha q,\beta,c;q;\frac{q^2}{\alpha\beta c}\\ q/\alpha,q^2/\beta,q^2/c,0\end{array}\right)=\frac{(q,q/\alpha\beta)_\infty}{(q/\alpha,q^2/\beta)_\infty}{}_3\phi_2\left(\begin{array}{c}\alpha q,\beta,-q/c;q;\frac{q}{\alpha\beta}\\ q^2/c,-q\end{array}\right),\tag{4.12}$$

where $|q^2/\alpha\beta c| < 1$, $|q/\alpha\beta| < 1$.
Then the theorem follows by using (3.8), (4.12) and the continued fraction (3.9). □

Theorem 4.4. *If α, β, z and q are real or complex parameters such that $|zq/\alpha\beta| < 1$, then*

$$ {}_2\psi_3\left(\begin{array}{c} \alpha, \beta; q; \frac{zq}{\alpha\beta} \\ q/\alpha, q/\beta, 0 \end{array}\right) = \frac{(q, q/\alpha\beta)_\infty}{(q/\alpha, q/\beta)_\infty} $$
$$ \left[1 + \frac{c_1(q/\alpha\beta)}{1} \, \frac{\frac{c_2}{c_1}(q/\alpha\beta)}{-\,1 + \frac{c_2}{c_1}(q/\alpha\beta) -} \cdots \frac{\frac{c_n}{c_{n-1}}(q/\alpha\beta)}{1 + \frac{c_n}{c_{n-1}}(q/\alpha\beta) -} \cdots \right], $$

where $c_0 = 1$ and

$$ \frac{c_n}{c_{n-1}} = \frac{(1-\alpha q^{n-1})(1-\beta q^{n-1})(1-zq^{n-1})(1-q^n/z)}{(1-q^{2n-1})(1-q^{2n})}, \qquad n = 1, 2, 3, ... $$

Proof. From the definition of basic hypergeometric series, we have

$$ {}_4\phi_3\left(\begin{array}{c} \alpha, \beta, z, q/z; q; \frac{q}{\alpha\beta} \\ -q, \sqrt{q}, -\sqrt{q} \end{array}\right) = \sum_{n=0}^{\infty} \frac{(\alpha, \beta, z, q/z; q)_n (q/\alpha\beta)^n}{(q, -q, \sqrt{q}, \sqrt{-q}; q)_n}, \qquad |q/\alpha\beta| < 1. \tag{4.13} $$

we choose $c_0 = 1$ and

$$ c_n = \frac{(\alpha, \beta, z, q/z; q)_n}{(q, -q, \sqrt{q}, \sqrt{-q}; q)_n}, \qquad n = 1, 2, 3... $$

then (4.13) becomes

$$ {}_4\phi_3\left(\begin{array}{c} \alpha, \beta, z, q/z; q; \frac{q}{\alpha\beta} \\ -q, \sqrt{q}, -\sqrt{q} \end{array}\right) = \sum_{k=0}^{\infty} c_k (\frac{q}{\alpha\beta})^k. \tag{4.14} $$

From Jones and Thron [3, eq(2.3.29), p. 37], a basic hypergeometric series can be expressed in terms of a continued fraction as,

$$ \sum_{k=0}^{n} c_k z^k = c_0 + \frac{c_1 z}{1-} \, \frac{\frac{c_2}{c_1} z}{1 + \frac{c_2}{c_1} z -} \, \frac{\frac{c_3}{c_2} z}{1 + \frac{c_3}{c_2} z -} \cdots \frac{\frac{c_n}{c_{n-1}} z}{1 + \frac{c_n}{c_{n-1}} z -} \cdots \tag{4.15} $$

we get the main result by using equations (4.14), (4.15) and (3.6). □

5. Special Cases

(1) Let $q \to 1$ in theorem (4.2) and using (2.3), we get

$$\frac{{}_4H_4\left(\begin{matrix}\alpha,\beta,c,d;(-1)\\ 1-\alpha,1-\beta,1-c,1-d\end{matrix}\right)}{\left(\begin{matrix}\alpha+1,\beta,c,d+1;(-1)\\ 1-\alpha,2-\beta,2-c,1-d\end{matrix}\right)} =$$

$$-\frac{1}{2}\left[1-\frac{A_0}{\frac{(-\alpha-d)}{(1-d)}+}\frac{B_0}{1}-\frac{A_1}{\frac{(-\alpha-d)}{(2-d)}+}\frac{B_1}{1}-\frac{A_2}{\frac{(-\alpha-d)}{(3-d)}+}\cdots\right], \quad (5.1)$$

where $Re(2-\alpha-\beta-c-d)>0$ and for $n=0,1,2,3,...$

$$A_n = \frac{(n+1-c-\alpha)(\beta+n)(n+1-c-d)}{(n+1-d)(2n+1-c)(2n+2-c)},$$

$$B_n = \frac{(n+1-\alpha)(n+2-\beta-c)(n+1+d)}{(n+1-d)(2n+2-c)(2n+3-c)}.$$

(2) Apply (3.3) to (5.1), we get

$$\frac{{}_5F_4\left(\begin{matrix}1,\alpha,\beta,c,d;(-1)\\ 1-\alpha,1-\beta,1-c,1-d\end{matrix}\right)+\frac{\alpha\beta cd}{(1-\alpha)(1-\beta)(1-c)(1-d)}{}_5F_4\left(\begin{matrix}1,\alpha+1,\beta+1,c+1,d+1;(-1)\\ 2-\alpha,2-\beta,2-c,2-d\end{matrix}\right)}{2\,{}_5F_4\left(\begin{matrix}1,\alpha+1,\beta,c,d+1;(-1)\\ 1-\alpha,2-\beta,2-c,1-d\end{matrix}\right)}$$

$$=-\frac{1}{2}\left[1-\frac{A_0}{\frac{(-\alpha-d)}{(1-d)}+}\frac{B_0}{1}-\frac{A_1}{\frac{(-\alpha-d)}{(2-d)}+}\frac{B_1}{1}-\frac{A_2}{\frac{(-\alpha-d)}{(3-d)}+}\cdots\right], \quad (5.2)$$

$$A_n = \frac{(n+1-c-\alpha)(\beta+n)(n+1-c-d)}{(n+1-d)(2n+1-c)(2n+2-c)},$$

$$B_n = \frac{(n+1-\alpha)(n+2-\beta-c)(n+1+d)}{(n+1-d)(2n+2-c)(2n+3-c)}, \quad n=0,1,2,3,...$$

(3) Let $d \to -\infty$ in (5.2), we get

$$\frac{{}_4F_3\left(\begin{matrix}1,\alpha,\beta,c,;1\\ 1-\alpha,1-\beta,1-c\end{matrix}\right)+\frac{\alpha\beta c}{(1-\alpha)(1-\beta)(1-c)}{}_4F_3\left(\begin{matrix}1,\alpha+1,\beta+1,c+1;1\\ 2-\alpha,2-\beta,2-c\end{matrix}\right)}{2\,{}_4F_3\left(\begin{matrix}1,\alpha+1,\beta,c;1\\ 1-\alpha,2-\beta,2-c\end{matrix}\right)}$$

$$=-\frac{1}{2}\left[1-\frac{A_0}{1+}\frac{B_0}{1-}\frac{A_1}{1+}\frac{B_1}{1-}\frac{A_2}{1+}\cdots\cdots\cdots\right], \quad (5.3)$$

where $Re(1-\alpha-\beta-c)>0$ and for n=0,1,2,3,...

$$A_n=\frac{(n+1-c-\alpha)(\beta+n)}{(2n+1-c)(2n+2-c)}, \qquad B_n=\frac{(\alpha-n-1)(n+2-\beta-c)}{(2n+2-c)(2n+3-c)}.$$

(4) put $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{2}$ in (5.3), we get

$$\frac{{}_2F_1\left(\begin{array}{c}1,c,;1\\ 1-c\end{array}\right)+\frac{c}{(1-c)}{}_2F_1\left(\begin{array}{c}1,c+1;1\\ 2-c\end{array}\right)}{2\,{}_2F_1\left(\begin{array}{c}1,c;1\\ 2-c\end{array}\right)}$$

$$=-\frac{1}{2}\left[1-\frac{A_0}{1+}\frac{B_0}{1-}\frac{A_1}{1+}\frac{B_1}{1-}\frac{A_2}{1+}.....\right],$$

provided $Re(-2c)>0$ and for $n=0,1,2,3,...$

$$A_n=\frac{(n+\frac{1}{2}-c)(n+\frac{1}{2})}{(2n+1-c)(2n+2-c)}, \qquad B_n=\frac{(-n-\frac{1}{2})(n+\frac{3}{2}-c)}{(2n+2-c)(2n+3-c)}.$$

(5) Take $\alpha\to\infty$ and $c\to\infty$ in theorem (4.2), we get

$$\frac{{}_2\psi_5\left(\begin{array}{c}\beta,d;q;\frac{q^2}{\beta d}\\ q/\beta,q/d,0,0,0\end{array}\right)}{{}_2\psi_5\left(\begin{array}{c}\beta,dq;q;\frac{q^3}{\beta d}\\ q^2/\beta,q/d,0,0,0\end{array}\right)}=\frac{1}{(1-q/\beta)}\left[1-\frac{A_0}{\frac{1}{(1-\frac{q}{d})}+}\frac{B_0}{1-}\frac{A_1}{\frac{1}{(1-\frac{q^2}{d})}+}\frac{B_1}{1-}...\right], \quad (5.4)$$

where $|q^2/\beta d|<1$ and for $n=0,1,2,3,...$

$$A_n=\frac{q^{n+1}(1-\beta q^n)}{\beta(1-q^{n+1}/d)}, \qquad B_n=\frac{-q^{n+1}(1-dq^{n+1})}{q(1-q^{n+1}/d)}.$$

(6) Let $q\to 1$ in (5.4) and using (2.3), we get

$$\frac{{}_2H_2\left(\begin{array}{c}\beta,d;(-1)\\ 1-\beta,1-d\end{array}\right)}{{}_2H_2\left(\begin{array}{c}\beta,d+1;(-1)\\ 2-\beta,1-d\end{array}\right)}=-\frac{1}{2}\left[1-\frac{A_0}{(\frac{-1}{2})_+}\frac{B_0}{1-}\frac{A_1}{(\frac{-1}{2})_+}\frac{B_1}{1-}\frac{A_2}{(\frac{-1}{2})_+}...\right],$$

where $Re(2-\beta-d)>0$ and for $n=0,1,2,3,...$

$$A_n=\frac{(n+\beta)}{(n+1-d)}, \qquad B_n=\frac{(n+1+d)}{(n+1-d)}.$$

(7) Put $q = q\alpha\beta$ in theorem (4.4), we get

$$ {}_2\psi_3\left(\begin{array}{c}\alpha,\beta;q;zq\\ \beta q,\alpha q,0\end{array}\right) = \frac{(q,q\alpha\beta)_\infty}{(\beta q,\alpha q)_\infty}\left[1+\frac{c_1 q}{1}-\frac{\frac{c_2}{c_1}q}{1+\frac{c_2}{c_1}q-}\cdots\frac{\frac{c_n}{c_{n-1}}q}{1+\frac{c_n}{c_{n-1}}q-}\cdots\right], \tag{5.5}$$

where $|q|<1$, $|z|<1$, $c_0=1$ and for $n=1,2,3,...$

$$\frac{c_n}{c_{n-1}} = \frac{(1-\alpha(\alpha\beta q)^{n-1})(1-\beta(\alpha\beta q)^{n-1})(1-z(\alpha\beta q)^{n-1})(1-(\alpha\beta q)^{n-1}/z)}{(1-(\alpha\beta q)^{2n-1})(1-(\alpha\beta q)^{2n})}.$$

(8) Let $q\to 1$ in (5.5) and using (2.3), we get

$$ {}_2H_2\left(\begin{array}{c}\alpha,\beta;(-z)\\ \beta+1,\alpha+1\end{array}\right)$$

$$= \frac{(\alpha+\beta+1)_\infty(2)_\infty}{(\beta+1,\alpha+1)_\infty}\left[1+\frac{c_1}{1}-\frac{\frac{c_2}{c_1}}{1+\frac{c_2}{c_1}-}\cdots\frac{\frac{c_n}{c_{n-1}}}{1+\frac{c_n}{c_{n-1}}-}\cdots\right], \tag{5.6}$$

where $|z|<1$, $c_0=1$ and for $n=1,2,3,....$

$$\frac{c_n}{c_{n-1}} = \frac{(n\alpha+(n-1)\beta+(n-1))((n-1)\alpha+n\beta+(n-1)(1-z)(1-1/z)}{2n(2n-1)(\alpha+\beta+1)^2}.$$

(9) Apply (3.3) and then substitute $z=z^2$ in (5.6), we get

$$ {}_3F_2\left(\begin{array}{c}1,\alpha,\beta;(-z^2)\\ \alpha+1,\beta+1\end{array}\right)+\frac{\alpha\beta}{(1-\alpha)(1-\beta)}{}_3F_2\left(\begin{array}{c}1,1-\alpha,1-\beta;(-1/z^2)\\ 2-\alpha,2-\beta\end{array}\right)$$

$$= \frac{(\alpha+\beta+1)_\infty(2)_\infty}{(\beta+1,\alpha+1)_\infty}\left[1+\frac{c_1}{1}-\frac{\frac{c_2}{c_1}}{1+\frac{c_2}{c_1}-}\frac{\frac{c_3}{c_2}}{1+\frac{c_3}{c_2}-}\cdots\frac{\frac{c_n}{c_{n-1}}}{1+\frac{c_n}{c_{n-1}}-}\cdots\right], \tag{5.7}$$

where $|z|<1$, $c_0=1$ and for $n=1,2,3,...$

$$\frac{c_n}{c_{n-1}} = \frac{(n\alpha+(n-1)\beta+(n-1))((n-1)\alpha+n\beta+(n-1)(1-z)(1-1/z)}{2n(2n-1)(\alpha+\beta+1)^2}.$$

(10) Put $\alpha=\beta=\frac{1}{2}$ in (5.7), we get

$$ {}_3F_2\left(\begin{array}{c}1,\frac{1}{2},\frac{1}{2};(-z^2)\\ \frac{3}{2},\frac{3}{2}\end{array}\right)+{}_3F_2\left(\begin{array}{c}1,\frac{1}{2},\frac{1}{2};(-1/z^2)\\ \frac{3}{2},\frac{3}{2}\end{array}\right)$$

$$= \frac{{(2)_\infty}^2}{{(\frac{3}{2})_\infty}^2}\left[1+\frac{c_1}{1}-\frac{\frac{c_2}{c_1}}{1+\frac{c_2}{c_1}-}\frac{\frac{c_3}{c_2}}{1+\frac{c_3}{c_2}-}\cdots\frac{\frac{c_n}{c_{n-1}}}{1+\frac{c_n}{c_{n-1}}-}\cdot\cdot\right], \tag{5.8}$$

where $|z|<1$, $c_0=1$ and for $n=1,2,3,...$

$$\frac{c_n}{c_{n-1}} = \frac{(2n-3/2)^2(1-z^2)(1-1/z^2)}{4(2n)(2n-1)}.$$

(11) substitute $x = z$ and $x = \frac{1}{z}$ in (3.4) then after some simplification (5.8) implies that

$$\frac{1}{z}\int \frac{tan^{-1}z}{z}dz - \int tan^{-1}(1/z)dz$$
$$= \frac{{(2)_\infty}^2}{{(\frac{3}{2})_\infty}^2}\left[1 + \frac{c_1}{1 -} \frac{\frac{c_2}{c_1}}{1 + \frac{c_2}{c_1} -} \frac{\frac{c_3}{c_2}}{1 + \frac{c_3}{c_2} -} \cdots \frac{\frac{c_n}{c_{n-1}}}{1 + \frac{c_n}{c_{n-1}} -} \cdot\cdot\right],$$

where $c_0 = 1$ and for $n = 1, 2, 3, ...$

$$\frac{c_n}{c_{n-1}} = \frac{(2n - 3/2)^2(1 - z^2)(1 - 1/z^2)}{4(2n)(2n - 1)}.$$

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UNIQUENESS OF RELAXED WEAKLY WEIGHTED SHARING OF DIFFERENTIAL-DIFFERENCE POLYNOMIALS OF ENTIRE FUNCTIONS

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Abstract. In this paper, we study the uniqueness of transcendental entire functions whose certain non-linear differential-difference polynomials share a small function with some weight in a relaxed manner. Our results generalize the earlier results of P. Sahoo and H. Karmakar [22].

1. Introduction

Let f be a nonconstant meromorphic function in the finite complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notations, definitions and basic results of Nevanlinna's value distribution theory (See [14], [23]).

A meromorphic function $\alpha = \alpha(z)$ is called a small function of f if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ is any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \longrightarrow \infty$ possibly outside of a set of finite linear measure.

Let f and g be two non-constant meromorphic functions in the finite complex plane $\mathbb{C}$. If for some complex number a (finite or infinite), $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, then we say that f and g share the value a CM (Counting multiplicities), and f and g share the value a IM (Ignoring multiplicities) whenever the multiplicities are ignored.

If k is a positive integer or infinity and $a \in \mathbb{C} \cup \infty$, then $E_{k)}(a; f)$ denotes the set of all zeros of $f - a$ with multiplicities not exceeding k, where

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zeros of $f-a$ is counted with multiplicities and $\overline{E}_{k)}(a;f)$ denotes the set of all distinct zeros of $f-a$ with multiplicities not exceeding k.

Let $N^E(r,a;f,g)$ $(\overline{N}^E(r,a;f,g))$ be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N^0(r,a;f,g)$ $(\overline{N}^0(r,a;f,g))$ be the counting function (reduced counting function) of all common zeros of $f-a$ and $g-a$ ignoring multiplicities.

If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}^E(r,a;f,g)=S(r,f)+S(r,g),$$

then we say that f and g share a "CM".

On the other hand, if

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}^0(r,a;f,g)=S(r,f)+S(r,g),$$

then we say that f and g share a "IM". We now require the following definitions.

We denote by $N(r,a;f\mid=1)$ the counting function of simple a points of f and $N(r,a;f\mid\leq k)$ $(N(r,a;f\mid\geq k))$ the counting function of those a points of f whose multiplicities are not greater(less) than k where each a point is counted according to its multiplicity.

$\overline{N}(r,a;f\mid\leq k)$ $(\overline{N}(r,a;f\mid\geq k))$are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f\mid<k)$, $(N(r,a;f\mid>k))$, $\overline{N}(r,a;f\mid<k)$, $(\overline{N}(r,a;f\mid>k))$ are defined analogously. Set

$$N_k(r,a;f)=\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geq 2)+\overline{N}(r,a;f\mid\geq 3)+\cdots+\overline{N}(r,a;f\mid\geq k).$$

To state the next results, we require the following definitions namely weakly weighted sharing and relaxed weighted sharing which are introduced by Lin and Lin [18] and Banerjee and Mukherjee [2] respectively.

The following notations can be found in [18].

Let f and g share a "IM" and k be a positive integer or ∞.

(i) $\overline{N}^E(r,a;f,g\mid\leq k)$ denotes the reduced counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k.

(ii) $\overline{N}^0(r,a;f,g\mid>k)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k.

Weakly weighted sharing [18]

Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. If

$$\begin{aligned}
\overline{N}(r, a, f \mid \leq k) - \overline{N}^{E}(r, a; f, g \mid \leq k) &= S(r, f),\\
\overline{N}(r, a, g \mid \leq k) - \overline{N}^{E}(r, a; f, g \mid \leq k) &= S(r, g),\\
\overline{N}(r, a, f \mid \geq k+1) - \overline{N}^{0}(r, a; f, g \mid \geq k+1) &= S(r, f),\\
\overline{N}(r, a, g \mid \geq k+1) - \overline{N}^{0}(r, a; f, g \mid \geq k+1) &= S(r, g),
\end{aligned}$$

or if $k = 0$ and

$$\begin{aligned}
\overline{N}(r, a, f) - \overline{N}^{0}(r, a; f, g) &= S(r, f),\\
\overline{N}(r, a, g) - \overline{N}^{0}(r, a; f, g) &= S(r, g),
\end{aligned}$$

then we say that f **and** g **weakly share the value** a **with weight** k and here we write f and g share "(a, k)".

In 2007, A. Banerjee and S. Mukherjee [2] introduced a new type of sharing known as relaxed weighted sharing, weaker than weakly weighted sharing which is defined as follows.

Definition 1.1. (**Relaxed weighted sharing**) [2] Let $a \in \mathbb{C} \cup \{\infty\}$ and k be a positive integer or infinity. Suppose that f and g share the value a "IM". If for $p \neq q$,

$$\sum_{p,q \leq k} \overline{N}(r, a; f \mid= p, g \mid= q) = S(r),$$

where $\overline{N}(r, a; f \mid= p; g \mid= q)$ the reduced counting function of common $a-$points of f and g with multiplicities p and q respectively, then we say that f and g share the value a with weight k in a relaxed manner and in that case we write f and g share $(a, k)^*$ to mean that f and g share a with weight k in a relaxed manner.

In the year 2015, P. Sahoo and H. Karmakar [22] obtained the following theorems and these are generalizations of earlier results of Zhang [25], Meng [20], Sahoo [21].

Theorem A. [22] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a non-zero complex constant, $n(\geq 1)$ and*

$m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 3k+2m+8$ when $m \leq k+1$ and $n \geq 6k-m+13$ when $m > k+1$. If $(f^n(z)(f(z)-1)^m f(z+\eta))^{(k)}$ and $(g^n(z)(g(z)-1)^m g(z+\eta))^{(k)}$ share $(\alpha(z), 2)^*$, then either $f(z) = g(z)$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1, w_2)$ is given by

$$R(w_1, w_2) = w_1^n (w_1 - 1)^m w_1(z+\eta) - w_2^n (w_2 - 1)^m w_2(z+\eta). \tag{1.1}$$

Theorem B. [22] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0, \infty)$ be a small function with respect to both $f(z)$ and $g(z)$. Suppose that η is a non-zero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 5k+4m+12$ when $m \leq k+1$ and $n \geq 10k-m+19$ when $m > k+1$. If $\overline{E}_{2)}\left(\alpha(z), (f^n(z)(f(z)-1)^m f(z+\eta))^{(k)}\right) = \overline{E}_{2)}\left(\alpha(z), (g^n(z)(g(z)-1)^m g(z+\eta))^{(k)}\right)$, then either $f(z) \equiv g(z)$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g) = 0$, where $R(w_1, w_2)$ is given by (1.1).*

We note that f and g share "(a,k)" means they share $(a,k)^*$ for $k \geq 1$ but not conversely.

Also from the definition of relaxed weighted sharing it is clear that for finite k, f and g share $(a,k)^*$ actually means they share a "IM" with some restrictions imposed on the common zeros of $f-a$ and $g-a$ up to multiplicity k. In particular if $k = 2$, the restrictions are minimum. In [1], A. Banerjee and S. Mukherjee shown that the lower bound of n can be significantly reduced by considering f and g share $(a,k)^*$ instead of $(a,0)$. Because of the above reasoning, in this paper we have considered relaxed weighted sharing instead of weakly weighted sharing and for more generalised differential-difference functions than the functions considered in the above mentioned Theorem A and Theorem B. That is, we have considered $\left(f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}\right)^{(k)}$ and $\left(g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j}\right)^{(k)}$ sharing a small function α with weight k in a relaxed manner.

The main results of this paper are as follows.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0, \infty)$ be a common small function with respect to $f(z)$ and $g(z)$, $c_j (j = 1, 2, \ldots, d)$ be distinct finite complex numbers and n, m and d*

are positive integers, k and $v_j(j=1,2,\ldots,d)$ are non-negative integers satisfying $n \geq 3k+2m+2\sigma+6$ when $m \leq k+1$ and $n \geq 6k-m+2\sigma+11$ when $m > k+1$, where $\sigma=\sum_{j=1}^{d} v_j$. If $\left(f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f\left(z+c_j\right)^{v_j}\right)^{(k)}$ and $\left(g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g\left(z+c_j\right)^{v_j}\right)^{(k)}$ share $(\alpha(z),2)^*$, then either $f(z) \equiv g(z)$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g)=0$, where $R\left(w_1,w_2\right)$ is given by

$$R\left(w_1,w_2\right)=w_1^n(w_1-1)^m \prod_{j=1}^{d} w_1\left(z+c_j\right)^{v_j}-w_2^n(w_2-1)^m \prod_{j=1}^{d} w_2\left(z+c_j\right)^{v_j}. \tag{1.2}$$

Remark 1.3. If $d=1$, $v_1=1$ and $c_1=\eta$ in Theorem 1.2, then Theorem 1.2 reduces to Theorem A.

Theorem 1.4. *Let $f(z)$ and $g(z)$ be transcendental entire functions of finite order, $\alpha(z)(\not\equiv 0,\infty)$ be a common small function with respect to $f(z)$ and $g(z)$, $c_j(j=1,2,\ldots,d)$ be distinct finite complex numbers and n, m and d are positive integers, k and $v_j(j=1,2,\ldots,d)$ are non-negative integers satisfying $n \geq 5k+4m+4\sigma+8$ when $m \leq k+1$ and $n \geq 10k-m+4\sigma+15$ when $m > k+1$, where $\sigma=\sum_{j=1}^{d} v_j$. If*

$$\begin{aligned}
&\overline{E}_{2)}\left(\alpha(z),\left(f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f\left(z+c_j\right)^{v_j}\right)^{(k)}\right)\\
=\ &\overline{E}_{2)}\left(\alpha(z),\left(g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g\left(z+c_j\right)^{v_j}\right)^{(k)}\right),
\end{aligned}$$

then the conclusion of Theorem 1.2 holds.

Remark 1.5. If $d=1$, $v_1=1$ and $c_1=\eta$ in Theorem 1.4, then Theorem 1.4 reduces to Theorem B.

2. The Lemmas

We need the following Lemmas to prove our results.
We denote by H the following function

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right), \tag{2.1}$$

where F and G are non-constant meromorphic functions defined in the complex plane $\mathbb{C}$.

Lemma 2.1. [24] *Let f be a non-constant meromorphic function and p, k be positive integers. Then*

$$N_p\left(r,\frac{1}{f^{(k)}}\right)=T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f). \tag{2.2}$$

$$N_p\left(r,\frac{1}{f^{(k)}}\right)\leq k\overline{N}(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f). \tag{2.3}$$

Lemma 2.2. [4] *Let $f(z)$ be a meromorphic function of order $\rho(f)<\infty$, and η be non-zero complex constant. Then, for each $\epsilon>0$, we have*

$$T(r,f(z+\eta))=T(r,f)+O\{r^{\rho(f)-1+\epsilon}\}+O(\log r).$$

Lemma 2.3. [3] *Let $f(z)$ be entire function of finite order and*

$$F=f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f\left(z+c_j\right)^{v_j}.$$

Then, $T(r,F)=(n+m+\sigma)T(r,f)+S(r,f)$, where $\sigma=\sum_{j=1}^{d}v_j$.

Lemma 2.4. [2] *Let F and G be non-constant meromorphic functions that share $(1,2)^*$ and $H\not\equiv 0$. Then*

$$\begin{aligned}T(r,F)\ &\leq\ N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+\overline{N}\left(r,\frac{1}{F}\right)\\&\quad+\overline{N}(r,F)-\ m\left(r,\frac{1}{F}\right)+S(r,F)+S(r,G)\end{aligned}$$

and the same inequality is true for $T(r,G)$.

Lemma 2.5. [19] *Let F and G be non-constant entire functions and $p\geq 2$ be an integer. If $\overline{E}_{p)}(1,F)=\overline{E}_{p)}(1,G)$ and $H\not\equiv 0$, then*

$$T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,F)+S(r,G)$$

and the same inequality is true for $T(r,G)$.

Lemma 2.6. *Let $f(z)$ and $g(z)$ be entire functions, $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$ be integers, and let*

$$F=\left(f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f\left(z+c_j\right)^{v_j}\right)^{(k)}$$

and

$$G = \left(g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g\left(z+c_j\right)^{v_j} \right)^{(k)}.$$

If there exist two non-zero constants c_1 and c_2 such that $\overline{N}\left(r, \frac{1}{F-c_1}\right) = \overline{N}\left(r, \frac{1}{G}\right)$ and $\overline{N}\left(r, \frac{1}{G-c_2}\right) = \overline{N}\left(r, \frac{1}{F}\right)$, then $n \leq 2k+m+\sigma+2$ when $m \leq k+1$ and $n \leq 4k-m+\sigma+4$ when $m > k+1$.

Proof. Let

$$F_1 = f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f\left(z+c_j\right)^{v_j}$$

and

$$G_1 = g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g\left(z+c_j\right)^{v_j}.$$

By Lemma 2.3, we have

$$T(r, F_1) = (n+m+\sigma)T(r,f) + S(r,f) \tag{2.4}$$

and

$$T(r, G_1) = (n+m+\sigma)T(r,g) + S(r,g). \tag{2.5}$$

Since F is an entire function and from Second fundamental theorem, we get

$$T(r,F) \leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-c_1}\right) + S(r,F). \tag{2.6}$$

From (2.2) - (2.4) and (2.6), we deduce

$$(n+m+\sigma)T(r,f) \leq N_{k+1}\left(r, \frac{1}{F_1}\right) + N_{k+1}\left(r, \frac{1}{G_1}\right) + S(r,f) + S(r,g).$$

That is,

$$\begin{aligned}(n+m+\sigma)T(r,f) \leq N_{k+1}&\left(r, \frac{1}{f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f\left(z+c_j\right)^{v_j}}\right) \\ &+ N_{k+1}\left(r, \frac{1}{g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g\left(z+c_j\right)^{v_j}}\right) \\ &+ S(r,f) + S(r,g).\end{aligned} \tag{2.7}$$

If $m \leq k+1$, then (2.7) reduces to

$$\begin{aligned}(n+m+\sigma)T(r,f) \leq{}& (k+1)T(r,f) + mT(r,f) + T\left(r, \prod_{j=1}^{d} f\left(z+c_j\right)^{v_j}\right)\\ &+ (k+1)T(r,g) + \ mT(r,g) + T\left(r, \prod_{j=1}^{d} g\left(z+c_j\right)^{v_j}\right)\\ &+ S(r,f) + S(r,g),\end{aligned}$$

which leads to

$$(n+m+\sigma)T(r,f) \leq (k+1+m+\sigma)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \tag{2.8}$$

Similarly, we get

$$(n+m+\sigma)T(r,g) \leq (k+1+m+\sigma)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \tag{2.9}$$

Combining (2.8) and (2.9), we get

$$(n+m+\sigma)(T(r,f)+T(r,g)) \leq 2(k+1+m+\sigma)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

this implies $n \leq 2k+m+\sigma+2$.
If $m > k+1$, then (2.7) reduces to

$$\begin{aligned}(n+m+\sigma)T(r,f) \leq{}& (k+1)T(r,f) + (k+1)T(r,f) + T\left(r, \prod_{j=1}^{d} f\left(z+c_j\right)^{v_j}\right)\\ &+ (k+1)T(r,g) + \ (k+1)T(r,g) + T\left(r, \prod_{j=1}^{d} g\left(z+c_j\right)^{v_j}\right)\\ &+ S(r,f) + S(r,g),\end{aligned}$$

which yields

$$(n+m+\sigma)T(r,f) \leq (2k+\sigma+2)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \tag{2.10}$$

Similarly, we get

$$(n+m+\sigma)T(r,g) \leq (2k+\sigma+2)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \tag{2.11}$$

Combining (2.10) and (2.11), we get

$$(n+m+\sigma)(T(r,f)+T(r,g)) \leq 2(2k+\sigma+2)(T(r,f)+T(r,g))+S(r,f)+S(r,g), \tag{2.12}$$

which gives that $n \leq 4k-m+\sigma+4$. □

3. Proofs of the Main Results

Proof of Theorem 1.2. Let $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$, where

$$F_1 = f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}$$

$$\text{and} \quad G_1 = g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j}.$$

Then F and G are transcendental meromorphic functions that share $(1,2)^*$ except for the zeros and poles of $\alpha(z)$.
Let $H \not\equiv 0$. Then, using Lemma 2.4 and since F and G are transcendental entire functions, we have

$$T(r,F) \le N_2\left(r,\frac{1}{F}\right) + N_2\left(r,\frac{1}{G}\right) + N_1\left(r,\frac{1}{F}\right) + S(r,F) + S(r,G). \quad (3.1)$$

From (2.2) and (2.3), (3.1) reduces to

$$T(r,F_1) \le N_{k+2}\left(r,\frac{1}{F_1}\right) + N_{k+2}\left(r,\frac{1}{G_1}\right) + N_{k+1}\left(r,\frac{1}{F_1}\right) + S(r,f) + S(r,g). \quad (3.2)$$

If $m \le k+1$, using (2.3) in (3.2) yields

$$\begin{aligned}(n+m+\sigma)T(r,f) \le\ & (k+2)T(r,f) + mT(r,f) + \sigma T(r,f) + (k+2)T(r,g) \\ & + mT(r,g) + \sigma T(r,g) + (k+1)T(r,f) + mT(r,f) \\ & + \sigma T(r,f) + S(r,f) + S(r,g),\end{aligned}$$

which implies that

$$(n+m+\sigma)T(r,f) \le (2k+2m+2\sigma+3)T(r,f) + (k+m+\sigma+2)T(r,g) + S(r,f) + S(r,g). \quad (3.3)$$

Similarly, we get

$$(n+m+\sigma)T(r,g) \le (2k+2m+2\sigma+3)T(r,g) + (k+m+\sigma+2)T(r,f) + S(r,f) + S(r,g). \quad (3.4)$$

Combining (3.3) and (3.4), we have

$$(n+m+\sigma)[T(r,f)+T(r,g)] \le (3k+3m+3\sigma+5)[T(r,f)+T(r,g)] + S(r,f) + S(r,g),$$

which is contradiction to $n \ge 3k+2m+2\sigma+6$.

If $m > k+1$, using (2.3), (3.2) can be reduced to

$$\begin{aligned}(n+m+\sigma)T(r,f) \leq (k+2)T(r,f) + (k+2)T(r,f) + \sigma T(r,f) + (k+2)T(r,g)\\ + (k+2)T(r,g) + \sigma T(r,g) + (k+1)T(r,f)\\ + (k+1)T(r,f) + \sigma T(r,f) + S(r,f) + S(r,g),\end{aligned}$$

which gives

$$(n+m+\sigma)T(r,f) \leq (4k+2\sigma+6)T(r,f)+(2k+\sigma+4)T(r,g)+S(r,f)+S(r,g). \tag{3.5}$$

Similarly, we get

$$(n+m+\sigma)T(r,g) \leq (4k+2\sigma+6)T(r,g)+(2k+\sigma+4)T(r,f)+S(r,f)+S(r,g). \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$(n+m+\sigma)[T(r,f)+T(r,g)] \leq (6k+3\sigma+10)[T(r,f)+T(r,g)]+S(r,f)+S(r,g),$$

which is contradiction to $n \geq 6k - m + 2\sigma + 11$.
Thus $H \equiv 0$. That is $\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) \equiv 0$, which implies $\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) \equiv \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$. Integrating both sides twice, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.7}$$

where $A(\neq 0)$ and B are constants.
From (3.7), it is clear that F, G share 1 CM and hence they share $(1,2)$.
We now discuss the following cases
Case 1: Let $B \neq 0$ and $A = B$. Then from (3.7), we get

$$\frac{1}{F-1} = \frac{BG}{G-1} \tag{3.8}$$

Subcase(i): If $B = -1$, then from (3.8), we have $FG = 1$, that is

$$\left(f^n(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}\right)^{(k)} \left(g^n(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j}\right)^{(k)} = \alpha^2(z), \tag{3.9}$$

which gives $N\left(r, \frac{1}{f}\right) = S(r,f)$ and $N\left(r, \frac{1}{f-1}\right) = S(r,f)$.
Since

$$\delta(0,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{N\left(r, \frac{1}{f}\right)}{T(r,f)} = 1,$$

$$\delta(1,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{N\left(r,\frac{1}{f-1}\right)}{T(r,f)} = 1,$$
$$\delta(\infty,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{N\left(r,f\right)}{T(r,f)} = 1,$$

we see that $\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3$, which is impossible.

Subcase(ii): If $B \neq -1$, then from (3.8) we deduce that $\frac{1}{F} = \frac{BG}{(B+1)G-1}$. Thus, $\overline{N}\left(r,\frac{1}{G-\frac{1}{B+1}}\right) = \overline{N}\left(r,\frac{1}{F}\right)$.

From Second Fundamental Theorem, we have

$$\begin{aligned} T(r,G) &\leq \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G-\frac{1}{B+1}}\right) + S(r,G) \\ &\leq \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{F}\right) + S(r,g). \end{aligned}$$

Hence,

$$T(r,G) \leq N_1\left(r,\frac{1}{G_1^{(k)}}\right) + N_1\left(r,\frac{1}{F_1^{(k)}}\right) + S(r,f) + S(r,g). \tag{3.10}$$

Using (2.2)-(2.4)in (3.10), we obtain that

$$\begin{aligned} (n+m+\sigma)T(r,g) \leq N_{k+1}&\left(r,\frac{1}{f^n(f(z)-1)^m\prod_{j=1}^{d} f\left(z+c_j\right)^{v_j}}\right) \\ +\ N_{k+1}&\left(r,\frac{1}{g^n(g(z)-1)^m\prod_{j=1}^{d} g\left(z+c_j\right)^{v_j}}\right) \\ +\ S(r,f)&+S(r,g). \end{aligned} \tag{3.11}$$

If $m \leq k+1$, then (3.11) gives

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \tag{3.12}$$

Similarly, we get

$$(n+m+\sigma)T(r,f) \leq (k+m+\sigma+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$(n+m+\sigma)(T(r,f)+T(r,g) \leq 2(k+m+\sigma+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which is contradiction to $n \geq 3k + 2m + 2\sigma + 6$.
On the other hand if $m > k+1$, then (3.11) gives

$$(n+m+\sigma)T(r,g) \leq (2k+\sigma+2)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \quad (3.14)$$

Similar to (3.14), we have

$$(n+m+\sigma)T(r,f) \leq (2k+\sigma+2)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \quad (3.15)$$

Combining (3.14) with (3.15), we get

$$(n+m+\sigma)(T(r,f)+T(r,g) \leq 2(2k+\sigma+2)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which contradicts to $n \geq 6k - m + 2\sigma + 11$.

Case 2: Let $B \neq 0$ and $A \neq B$. Then from (3.7), we get

$$F = \frac{(1+B)G - (B-A+1)}{BG + (A-B)},$$

which implies that $\overline{N}\left(r, \frac{1}{G - \frac{(B-A+1)}{1+B}}\right) = \overline{N}\left(r, \frac{1}{F}\right)$.
Proceeding in a manner similar to Subcase (ii) of Case 1, we get a contradiction.
Case 3: Let $B = 0$. Then from (3.7), we get immediately that $G = AF - (A-1)$.
Subcase (i): If $A \neq 1$, then $G = AF - (A-1)$ follows that
$\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right)$ and $\overline{N}\left(r, \frac{1}{G+(A-1)}\right) = \overline{N}\left(r, \frac{1}{F}\right)$.
Applying Lemma 2.6, we arrive at a contradiction.
Subcase(ii): $A = 1$ gives $F = G$, that is

$$\left(f^n (f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}\right)^{(k)} = \left(g^n (g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j}\right)^{(k)}.$$

Integrating once, we obtain that

$$\left(f^n (f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}\right)^{(k-1)} = \left(g^n (g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j}\right)^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant.

If $c_{k-1} \neq 0$, then Lemma 2.6 follows that $n \leq 2k+m+\sigma$ when $m \leq k+1$ and $n \leq 4k - m + \sigma$ when $m > k+1$, a contradiction to the hypothesis.

Hence $c_{k-1}=0$.
Repeating the process k times, we deduce that

$$f^{n}\left(f(z)-1\right)^{m}\prod_{j=1}^{d}f\left(z+c_{j}\right)^{v_{j}}=g^{n}\left(g(z)-1\right)^{m}\prod_{j=1}^{d}g\left(z+c_{j}\right)^{v_{j}}. \tag{3.16}$$

Let $h=\frac{f}{g}$.
If $h(z)$ is a constant, let $h=t$, then substituting $f=gt$ in (3.16), we get

$$t^{n}g^{n}(z)\left(tg(z)-1\right)^{m}\prod_{j=1}^{d}t^{\sigma}g\left(z+c_{j}\right)^{v_{j}}=g^{n}(z)\left(g(z)-1\right)^{m}\prod_{j=1}^{d}g\left(z+c_{j}\right)^{v_{j}}.$$

Simplifying we get that

$$t^{n+\sigma}\left(tg(z)-1\right)^{m}=\left(g(z)-1\right)^{m},$$

which leads to

$$\begin{aligned}&t^{n+\sigma}\left(t^{m}g^{m}(z)-mC_{1}t^{m-1}g^{m-1}(z)-,\cdots,-1\right)\\&=\left(g^{m}(z)-mC_{1}g^{m-1}(z)-,\cdots,-1\right),\end{aligned}$$

which implies $t^{n+\sigma+m}=t^{n+\sigma+m-1}=\ldots=t^{n+\sigma}=1$, that is $t=1$. Thus $f(z)\equiv g(z)$.

If $h(z)$ is not a constant, then from (3.16) f and g satisfy the algebraic equation $R(f,g)=0$, where $R\left(w_{1},w_{2}\right)$ is given by

$$R\left(w_{1},w_{2}\right)=w_{1}^{n}(w_{1}-1)^{m}\prod_{j=1}^{d}w_{1}\left(z+c_{j}\right)^{v_{j}}-w_{2}^{n}(w_{2}-1)^{m}\prod_{j=1}^{d}w_{2}\left(z+c_{j}\right)^{v_{j}}.$$

□

Proof of Theorem 1.4. Let F, G, F_1, G_1 be as in Theorem 1.1. Then F and G are transcendental meromorphic functions such that $\overline{E}_{2)}(1,F)=\overline{E}_{2)}(1,G)$ except for the zeros and poles of $\alpha(z)$.
Let $H\not\equiv 0$. Then, from Lemma 2.5, (2.2) - (2.4), we deduce that

$$\begin{aligned}(n+m+\sigma)T(r,f)\leq\ & N_{k+2}\left(r,\frac{1}{F_{1}}\right)+N_{k+2}\left(r,\frac{1}{G_{1}}\right)+2N_{k+1}\left(r,\frac{1}{F_{1}}\right)\\&+\ N_{k+1}\left(r,\frac{1}{G_{1}}\right)+S(r,f)+S(r,g).\end{aligned} \tag{3.17}$$

If $m \leq k+1$, then (3.17) gives

$$(n+m+\sigma)T(r,f) \leq (3k+3m+3\sigma+4)T(r,f) + (2k+2m+2\sigma+3)T(r,g)+S(r,f)+S(r,g). \tag{3.18}$$

Similarly, we get

$$(n+m+\sigma)T(r,g) \leq (3k+3m+3\sigma+4)T(r,g) + (2k+2m+2\sigma+3)T(r,f)+S(r,f)+S(r,g). \tag{3.19}$$

Combining (3.18) and (3.19), we get

$$(n+m+\sigma)(T(r,f)+T(r,g)) \leq (5k+5m+5\sigma+7)(T(r,f)+T(r,g)) + S(r,f)+S(r,g),$$

which is contradiction to $n \geq 5k+4m+4\sigma+8$.

If $m > k+1$, then from (3.17), we deduce that

$$(n+m+\sigma)T(r,f) \leq (6k+3\sigma+8)T(r,f)+(4k+2\sigma+6)T(r,g)+S(r,f)+S(r,g). \tag{3.20}$$

Similarly, we get

$$(n+m+\sigma)T(r,g) \leq (6k+3\sigma+8)T(r,g)+(4k+2\sigma+6)T(r,f)+S(r,f)+S(r,g). \tag{3.21}$$

Combining (3.20) and (3.21), we get

$$(n+m+\sigma)(T(r,f)+T(r,g)) \leq (10k+5\sigma+14)(T(r,f)+T(r,g))+S(r,f)+S(r,g),$$

which is contradiction to $n \geq 10k-m+5\sigma+15$. Thus $H \equiv 0$, and the remaining proof of the Theorem 1.4 follows in the same lines as in the proof of the Theorem 1.2. $\square$

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LOXODROMES ON THE PLANE

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Abstract. In this paper, we discuss some very interesting curves on the plane: equiangular spirals and the closely related planar loxodromes. The planar loxodromes show the streamlines of a flat fluid with one sink and one source. As we will see, some of these curve are very curious. The paper uses elementary facts from the theory of complex variables and vector analysis.

1. Introduction

By definition, loxodromes are curves on the sphere that cross all meridians at the same angle, Figure 1. They are also called rhumb lines – see [1, 6], and are closely related to navigation paths and the Mercator projection.

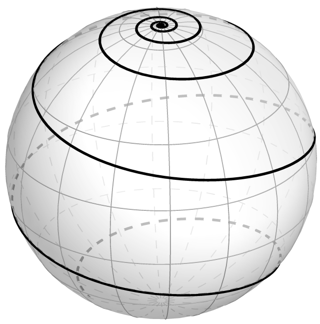

Figure 1

2010 Mathematics Subject Classification: 26B20, 30B10, 53A04, 76B47
Key words and phrases: Equiangular spirals, loxodromes, planar loxodromes, complex numbers, velocity vector field, fluid streamlines, sinks and sources, dipoles.

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The stereographic projection transforms such curves into equiangular spirals. These are spirals in the plane that cut all rays from the origin at a constant angle (Figure 2). The popular equiangular spiral, also called logarithmic spiral, has polar equation

$$r(\theta) = r_0 e^{k\theta}, -\infty < \theta < +\infty. \tag{1.1}$$

For more information about this spiral see [2, 3].

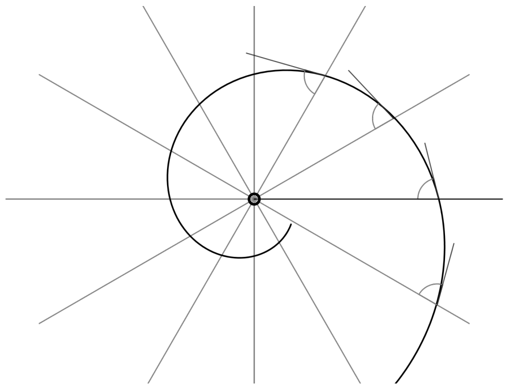

FIGURE 2

In this article we will discuss special curves on the plane which are also called loxodromes and are images of equiangular spirals by the Möbius transformation. These are the *planar loxodromes.* Planar loxodromes can be used to model streamlines of fluid flows with sinks and sources.
First we recall the simple model of a vortex from [3] and then we will show how planar loxodromes are related to equiangular spirals. In Section 2 (The Dipole) the reader will see the explicit equations of some very interesting planar loxodromes.

2. VECTOR FIELDS ON THE PLANE

We consider a flat fluid on the plane which moves according to a velocity vector field $V(x, y)$ [7]. For convenience we identify the xy-plane with the complex z-plane, by setting $z = x + iy = (x, y)$. Vectors in standard position (starting from the origin), and complex numbers are also identified by the equation $< u, v > = u + iv$. With this agreement we can write the vector field in several forms depending on the needs

$$V(z) = V(x, y) =< u, v >= u + iv \tag{2.1}$$

First, we assume that V is defined everywhere except, possibly, at the origin. Every moving point (particle) $M(x, y)$ in our fluid has velocity $V =< u(x, y), v(x, y) >$.The points governed by this velocity vector field move on streamlines. A streamline (trajectory) is a curve, at each point of which the velocity vector $< u, v >$ is tangent to it. If the smooth curve L is defined by the parametric equations

$$x = x(t),\ y = y(t),\ t_1 \leq t \leq t_2, \tag{2.2}$$

then L is a streamline for the velocity vector field (2.1), if and only if the tangent vector $< x\prime, y\prime >$ and the velocity vector $< u, v >$ are parallel at every point (x, y) on L.

We will use some definitions from vector calculus. Let G be a closed, positively oriented (counterclockwise) smooth curve surrounding a convex domain D. We can think that D is a disk. The integral

$$Cir(V : G) = \int_G V\ dr = \int_G u\,dx + v\,dy, \tag{2.3}$$

(where $dr =< dx, dy >$) defines the *circulation* of the field along the curve. Nonzero circulation indicates the presence of whirls inside D. We also define the *flux* through the curve G

$$Flux(V : G) = \int_G u\,dy - v\,dx. \tag{2.4}$$

Nonzero flux indicates the appearance or disappearance of fluid inside D, i.e. the presence of sources or sinks. We formally define now *vortex, source* and *sink*.

Definition 2.1. A point M is called a *vortex* for the vector field V, if there is a neighborhood U of M such that the circulation $Cir(V : G)$ on any circle $G \subset U$ centered at M is nonzero. Shortly, at a vortex the fluid spins, not necessarily appearing or disappearing.

Definition 2.2. With M and U as above, the point M is called a *source*, if $Flux(V : G) > 0$ and a *sink*, if $Flux(V : G) < 0$ for every circle $G \subset U$ centered at M.

In these definitions we do not require M to be in the domain of the field. It may be a singular point.
Before proceeding further, recall that multiplying a complex number z by the exponential $e^{i\alpha}$ produces a counterclockwise rotation about the origin by angle α.
When considering a flat fluid with possibly one source/sink, it is reasonable to assume that the fluid is described by a two-dimensional velocity vector field V with possibly one singularity at the origin, which could be either a sink, a source, or neither, and there are no other sinks or sources except O. We also do not permit the presence of any whirls which are not centered at O . It is natural to assume that this field is rotationally invariant according to O. Here are the exact conditions:

(A) The vector field $V = u + iv$ is smooth, i.e. u, v have continuous partial derivatives everywhere except, possibly, at the origin O.
(B) The vector field has no sinks or sources anywhere except, possibly, at the origin O, and also no whirls. This means both integrals (2.3), (2.4) are zero for every closed curve whose interior is separated from O.
(C) V is rotationally invariant, i.e. $e^{i\alpha}V(z) = V(e^{i\alpha}z)$ for all $z \neq 0, \alpha \in (0, \pi)$. When we rotate the plane about the origin at angle α, the picture we see does not change.

Such fields are characterized by the following theorem which was proved in [3].

3. The Spiral Vortex Theorem

Theorem 3.1. ***Part 1.*** *A vector field V satisfies (A), (B) and (C), if and only if it has the form*

$$V(z) = c/\bar{z} \tag{3.1}$$

where $c = a + ib$ is a complex constant.
Part 2. *When $b \neq 0$, the streamlines of this field are equiangular spirals with polar equation*

$$r = r_0\, e^{kt}, \ -\infty < t < \infty. \tag{3.2}$$

Here r_0 is an arbitrary real constant and $k = a/b$. When $b = 0$, the streamlines are rays going from O to infinity. When $a = 0$ the streamlines

are concentric circles centered at the origin [3].
We will give now an independent proof of Part 2, different from the one in [3].

Proof. We will show that the streamlines for (3.1) are equiangular spirals. From (3.1), the function $\overline{V(z)} = u - iv = \frac{a-ib}{z}$ is holomorphic and has antiderivative $f(z) = (a - ib)\log z$ in any simply connected domain that excludes the origin. This is also called a potential for the vector field. Let $Ref = \varphi(x, y)$, $Imf = \psi(x, y)$, so that $f(z) = \varphi + i\psi$. We claim that the streamlines of the vector field $V = u + iv$ are given by the family of Cartesian equations

$$\psi(x, y) = C \tag{3.3}$$

where C is an arbitrary constant. Indeed, let L be a stream line with parametric equations $x = x(t)$, $y = y(t)$. Then the vector $< x', y' >$ is parallel to the vector $< u, v >$ at every point on L. At the same time we have from the Cauchy-Riemann equations

$$f'(z) = \varphi_x + i\psi_x = \psi_y - i\varphi_x = u - iv$$

so that $< \psi_y, -\psi_x > = < u, v >$. We conclude that the vectors $< x', y' >$ and $< \psi_y, -\psi_x >$ are parallel. But then the vectors $< \psi_x,\ \psi_y >$ and $< x', y' >$ are orthogonal at each point (x, y), because $< \psi_x,\ \psi_y >$ and $< \psi_y, -\psi_x >$ are orthogonal. We have for every t

$$\frac{d}{dt}\psi(x(t),\ y(t)) = \psi_x x'(t) + \psi_y y'(t) = 0$$

and therefore, $\psi(x(t),\ y(t)) = \text{constant}$.
In our case $\psi = Im\{(a - ib)\log z\} = Im\{(a - ib)(\ln|z| + i\arg(z)\}$. Set $\arg(z) = t$ and $|z| = r$. Then equation (3.3) becomes

$$a\theta - b\ln r = C. \tag{3.4}$$

When $b \neq 0$, we apply the natural exponential function to both sides to get the polar equation $r = r_0 e^{kt}$, $k = a/b$, where the parameter t is identified with the polar angle and we allow it to run from $-\infty$ to $+\infty$. This is exactly the polar equation of the equiangular spiral. When $b = 0$ we have from (3.4) the family of polar equations $\theta = C/a$. These are rays from the origin to infinity. When $b \neq 0$ but $a = 0$, then $k = 0$ and we have concentric circles $r = r_0$ centered at the origin. □

The parametric equations of the spiral (3.2) are given by

$$x = r_0 e^{kt} \cos t, \ y = r_0 e^{kt} \sin t, -\infty < t < +\infty. \tag{3.5}$$

For the complex number $z = x + iy$ describing the spiral we have

$$z = r_0 e^{kt} e^{it} = r_0 e^{(k+i)t}. \tag{3.6}$$

4. The Dipole

The vortex described above can be a source or a sink in a flat fluid. Suppose we have a source at the origin. Then, as the streamlines show, we have a sink at infinity. After all, the fluid cannot increase indefinitely, it has to go somewhere! This interpretation helps to address the case of two finite singularities, say, one source at $z = 1$ and one sink at $z = -1$. The Möbius transformation

$$w = \frac{1-z}{1+z} \tag{4.1}$$

brings 0 to 1 and ∞ to -1. All straight lines through 0 (and ∞) turn into circles through $z = 1$ and $z = -1$, as shown on Figure 3.

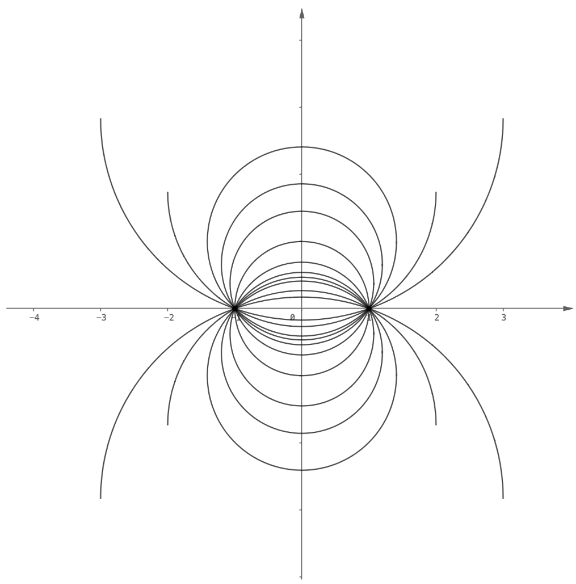

Figure 3

All circles centered at 0 turn into circles perpendicular to them, Figure 4.

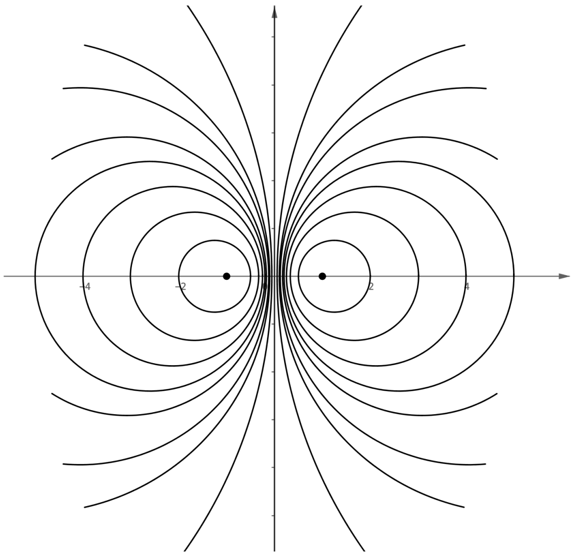

FIGURE 4

Suppose we have one equiangular spiral with pole 0 which cuts all straight lines through this pole at angle α. The images of these lined by (4.1) are circles through $z = 1$ and $z = -1$. The transformation (4.1) is conformal and preserves angles, so the image of that spiral will be a spiral line winding endlessly around the two poles at 1 and -1, and cutting all the circles through them by the same angle α. Such a curve is called a planar loxodrome [4], or double spiral [5, p. 187]. If a logarithmic spiral is defined by the polar equation (3.2), the corresponding double spiral has parametric equations:

$$\begin{aligned} x(t) &= \frac{1 - r_0^2 e^{2kt}}{1 + 2r_0 e^{kt}\cos t + r_0^2 e^{2kt}} \\ y(t) &= \frac{-2r_0 e^{kt}\sin t}{1 + 2r_0 e^{kt}\cos t + r_0^2 e^{2kt}} \end{aligned} \tag{4.2}$$

where $-\infty < t < \infty$. On Figure 5 we show a loxodrome cutting all circles from Figure 3 at the same angle, say, α. This is a streamline of the vector field with singularities at $z = -1$ and $z = 1$.

Next we show a family of loxodromes obtained from the equations (4.2) with $k = 0.5$ and several different values of r_0:

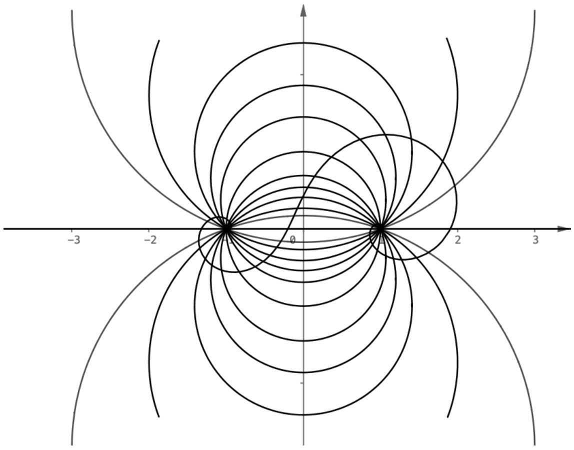

FIGURE 5

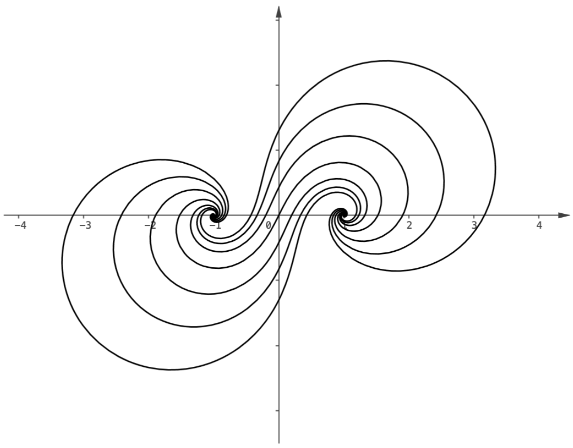

FIGURE 6

When angle $\alpha = 0$, the streamlines are all the circles on Figure 3. When $\alpha = \pi/2$, the streamlines are the circles on Figure 4.

Interesting symmetry! When $k = 0.5$ and $r_0 = 1$ equations (4.2) become

$$x(t) = \frac{1 - e^t}{1 + 2e^{t/2}\cos t + e^t}, \;\; y(t) = \frac{-2e^{t/2}\sin t}{1 + 2e^{t/2}\cos t + e^t} \qquad (4.3)$$

with the property $x(-t) = -x(t),\ y(-t) = -y(t)$, so changing t to $-t$ we go from the point $(x(t), y(t))$ to the symmetrical (with respect to the origin) point $(-x(t), -y(t))$. This special symmetrical loxodrome with equations (4.3) is shown on Figure 7. Swapping the poles changes only the direction on it.

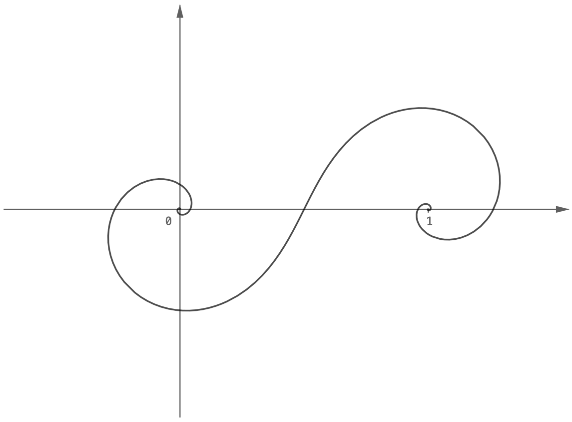

FIGURE 7

However, if we draw the streamline for the same $k = 0.5$, but $r_0 = 3$ (Figure 8) we see something different.

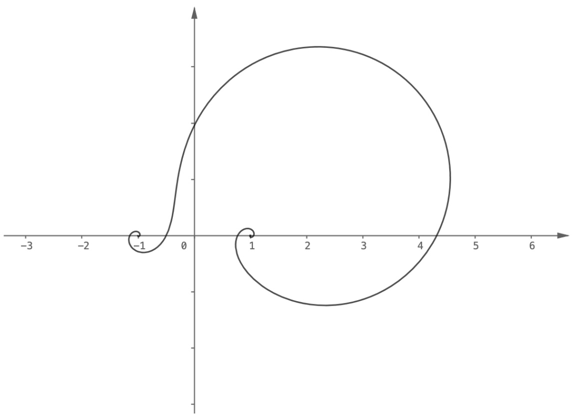

FIGURE 8

What happened to the symmetry? We expect that interchanging the poles does not change the shape of the streamlines. Well, there is no mistake! The symmetry exists, but on a different level. The streamline on Figure 8 has a symmetric partner: the streamline corresponding to $k = 0.5,\ r_0 = 1/3$. They both constitute the symmetrical pair shown on Figure 9.

Symmetrical pairs of double spirals can be found in Figure 6, they correspond to pairs r_0 and $1/r_0$. Using the representation (3.6) and equation

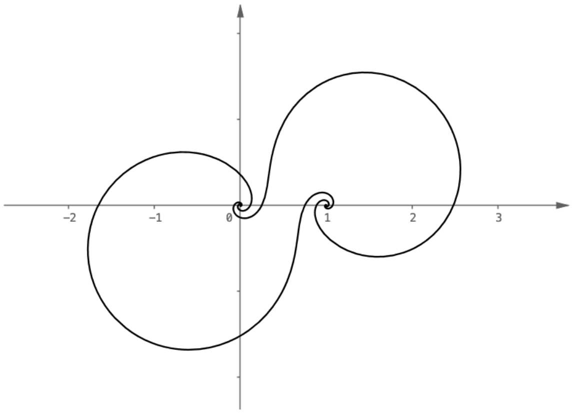

FIGURE 9

(4.1) we can explain the symmetry by writing (4.1) in the form

$$\frac{1-r_0e^{(k+i)t}}{1+r_0e^{(k+i)t}} = -\frac{1-(1/r_0)\,e^{-(k+i)t}}{1+(1/r_0)\,e^{-(k+i)t}} \tag{4.4}$$

More surprises. Among all loxodromes with poles -1 and 1 there is one passing through... infinity! Indeed, this is the image by (4.1) of the equiangular spiral in the z-plane defined by (3.2) which passes through $z = -1$ (that is, the point $(-1,0)$) . When $k = 1/2$ and $r_0 = e^{-\pi/2}$, the equiangular spiral with polar equation $r = e^{-\pi/2}e^{t/2}$ reaches the point $z = 1$ for $t = \pi$. The loxodrome defined by (4.2) with $k = 1/2$ and $r_0 = e^{-\pi/2}$ goes through infinity for $t = \pi$ and has an asymptote $y = -2x$. This unique loxodrome can be seen on Figure 10 together with the symmetrical loxodrome from Figure 7 and the asymptote. These three curves constitute an interesting composition.

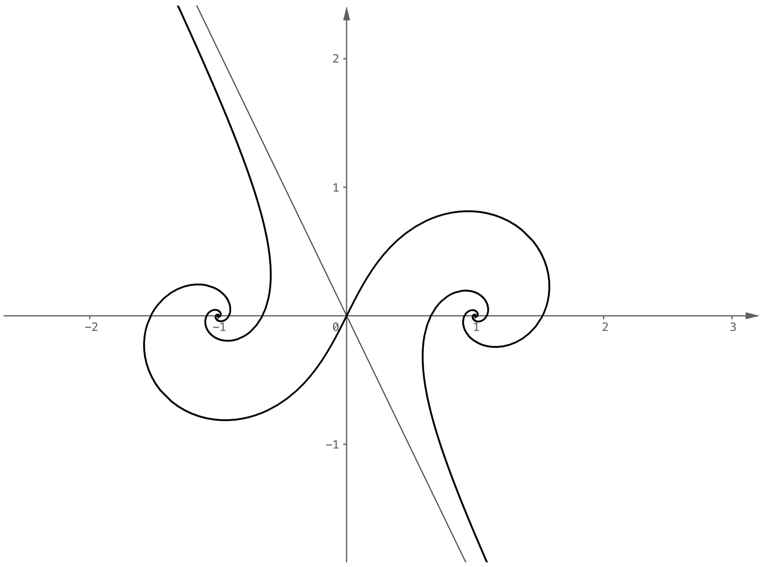

FIGURE 10

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PROBLEM SECTION

In Volume 91 (3-4) 2022 of The Mathematics Student, we had invited solutions from the readers to the Problems 2, 3, 4 and 5 mentioned in MS 90 (3-4) 2021, solutions to problems 1, 4, 5, 7, 8, 9 and 12 of MS 91 (1-2) 2022 as well as solutions to the eight new problems, till January 10, 2023.

As regards to solutions to the four Problems mentioned in MS 90 (3-4) 2021, we received one correct solution to Problem 3 by Mr. Andrés Ventas (Spain). This solution is presented in this section. No solutions were received to any of the remaining three problems from the readers of the Math. Student, so we will print solutions provided by the proposers of the problems. Out of the seven problems of MS 91 (1-2) 2022, we received solutions to problems 1, 4, 7, 9 and 12 which will be presented in this section. No solutions were received to problems 5 and 8, so the solutions provided by the proposer of the problems are being printed in the section.

As far as solutions to the eight new problems mentioned in MS 91 (3-4) 2022 are concerned, we received solutions to problems 1, 2, 3, 5, 6, 7 and 8 by the readers. These solutions are being presented in this section.

We pose **eight new Problems** in this section. We invite solutions from the readers of the Math. Student to Problem 4 of MS 91 (3-4) 2022 and solutions to the eight new problems till July 31, 2023. Correct solutions received from the readers by this date will be published in Volume 92 (3-4) 2023 of The Mathematics Student. This volume is scheduled to be published in August 2023.

New Problems

Prof. B. Sury, Indian Statistical Institute, Bangalore proposed the following four problems.

MS 92 (1-2) 2023 : Problem 1. Suppose a cube hanging on a vertex is immersed slowly into a tank of water as in the figure. As it enters the water, the shape of the outline of water on the cube changes from a point to a triangle etc. When it is half submerged, what is the shape?

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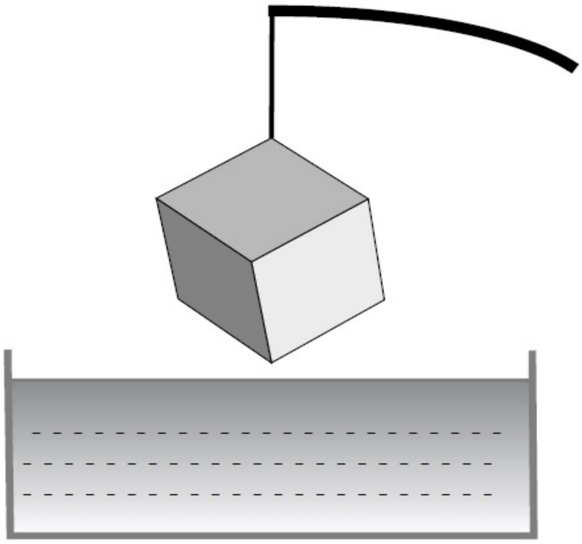
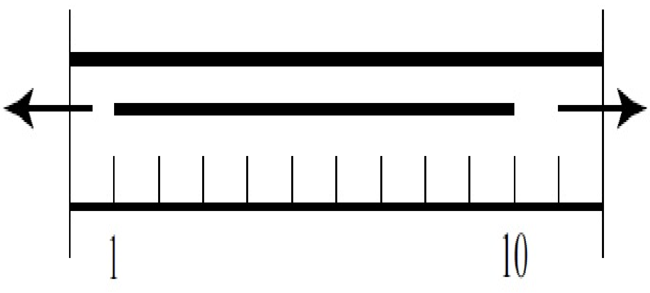


MS 92 (1-2) 2023 : Problem 2.
(a) If two points are randomly selected from the interval $[0, 1]$, what is the average distance between them?
(b) If two points are randomly selected from the interval $[0, 1]$, what is the probability that the distance between them is less than $1/2$?

MS 92 (1-2) 2023 : Problem 3. Consider a ruler which is 12 inches long. A rubber band before stretching covers 9 inches, say from the 1 mark to the 10 inch mark as in the figure below. The rubber band is stretched both sides so as to cover the ruler completely. Assume that during the stretching, the points near the left end move to the left and those towards the right end move to the right. Then, there is a point in between where a point of the rubber band ends up where it started from. Find that point.



There was a mistake in Problem 2 of MS 90 (3-4) 2021. The correct problem is produced here.

MS 92 (1-2) 2023 : Problem 4. Call a positive integer N 'powerful' if it can be written as $a^3 + b^5 + c^7 + d^9 + e^{11}$ for positive integers a, b, c, d, e. Show that there exist arbitrarily large numbers that are NOT powerful.

Dr. Anup Dixit, Institute of Mathematical Sciences, Chennai proposed the following two problems.

MS 92 (1-2) 2023 : Problem 5. Let $x_n = \min\{|a - b\sqrt{7}| :$ where $a, b \in \mathbb{Z}^+, a + b = n\}$. Find $\sup_n x_n$.

MS 92 (1-2) 2023 : Problem 6. For an odd positive integer m, let $\tilde{\phi}(m)$ denote the number of positive integers $r < m/4$ such that $gcd(r, m) = 1$. Show that $\tilde{\phi}(m)$ is odd if and only if $m = p^k$ for k a positive integer and $p \equiv 5, 7 \bmod 8$.

Prof. Shpetim Rexhepi and Ilir Demiri, Mother Teresa University, Skopje, North Macedonia proposed the following two problems.

MS 92 (1-2) 2023 : Problem 7. Prove the identity

$$\int_0^\infty \frac{xe^x}{(tanh(\frac{x}{2}) + tanh^2(\frac{x}{2}))(e^x + 1)^2}\, dx$$

$= \frac{\pi^2}{12}$

MS 92 (1-2) 2023 : Problem 8. Prove that, for $a, b, c \in \mathbb{R}^+$ and $abc = 1$, inequality

$$\left(a^3 + 1\right)\left(b^3 + 1\right)\left(c^3 + 1\right) \geq \left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right)$$

holds.

Solutions to the Old Problems

MS 90 (3-4) 2021 : Problem 3 (Proposed by Prof. B. Sury).
Let p be a prime and, for each $1 \leq i \leq p - 1$, let $i^p \equiv a_i \bmod p^2$ where $0 < a_i < p^2$. Find the value of $\sum_{i=1}^{p-1} a_i$.

Andrés Ventas, Santiago de Compostela, Galicia, Spain provided a correct solution to the problem. However, the following solution given by Prof. Sury is more elegant.

Solution (by Prof. Sury).

We consider p odd; for $p = 2$, the sum is 1.
Denoting $s = \sum_i a_i$ and adding it with itself written in the opposite order, we get

$$2s = \sum_{i=1}^{p-1} (a_i + a_{p-i}).$$

Now,

$$a_i + a_{p-i} \equiv i^p + (p-i)^p = i^p + \sum_{r=0}^{p} \binom{p}{r} p^r i^{p-r}.$$

The terms i^p and $-i^p$ cancel and all the other terms are divisible by p^2. Hence,

$$a_i + a_{p-i} \equiv 0 \ (mod \ p^2).$$

However, each $a_j < p^2$ which gives $a_i + a_{p-i} < 2p^2$. Thus, $a_i + a_{p-i} = p^2$ for each $i \leq p-1$. In other words, $2s = (p-1)p^2$ which gives $s = \frac{p^3 - p^2}{2}$.

MS 90 (3-4) 2023 : Problem 4. (Proposed by Prof. Sury)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined as $f(x_1, \cdots, x_n) = \frac{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}{x_1^{b_1} + \cdots + x_n^{b_n}}$ for $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$ and $f(0, \cdots, 0) = 0$, where a_i's are positive integers and b_i's are even positive integers. Determine precisely all the a_i's and b_i's for which f is discontinuous at 0.

Solution (by Prof. Sury).

Writing $b_1 \leq b_2 \leq \cdots \leq b_n$, we claim that f is discontinuous if and only if $\sum_{i=1}^{n} \frac{a_i}{b_i} \leq 1$.
Write $c = \sum_{i=1}^{n} \frac{a_i}{b_i}$ and $g = x_1^{b_1} + \cdots + x_n^{b_n}$. Then

$$f = \frac{1}{g^{1-c}} \frac{x_1^{a_1}}{g^{a_1/b_1}} \cdots \frac{x_n^{a_n}}{g^{a_n/b_n}} = g^{c-1} \prod_{i=1}^{n} \frac{x_i^{a_i}}{g^{a_i/b_i}}.$$

If $c \leq 1$, then for any $t > 0$,

$$f(t^{1/b_1}, \cdots, t^{1/b_n}) = \frac{t^c}{nt} = \frac{t^{c-1}}{n}.$$

Therefore, when $c \leq 1$, the above does not tend to $f(0, \cdots, 0) = 0$ which means f is discontinuous at 0.
Conversely, suppose $c > 1$. As

$$\left| \frac{x_i^{a_i}}{g^{a_i/b_i}} \right| \leq \left| \frac{x_i^{a_i}}{(x_i^{b_i})^{a_i/b_i}} \right| = 1,$$

we have

$$|f(x_1, \cdots, x_n)| \leq g^{c-1}.$$

The RHS above tends to 0 as $(x_1, \cdots, x_n) \to 0$ because $c > 1$. Hence, f is continuous at 0.

MS 90 (3-4) 2023 : Problem 5 (Proposed by Prof. Sury)
Let S be a finite set of points in the plane and suppose each of the points is given one of two colours red or blue. Prove that there exist $P, Q \in S$ such that all the points on the line joining P and Q have the same colour.
Remark. It should also be mentioned in the problem that not all points of S are collinear.

Solution (by Prof. Sury).

We give the values 1 and -1 to red and blue respectively say. We use the following statement usually known as projective duality: if we have n lines in the projective plane which do not all pass through a point, there must exist a point that lies on exactly two of the lines. In fact, the problem is equivalent to the following assertion using projective duality:
If we have n great circles on a sphere, each one given the value 1 or -1, and not all passing through the same point, there must exist a point that lies on at least two of the circles such that all circles passing through that point have the same value.
The circles determine a spherical polyhedron with edges labelled by 1 and -1. Let us assume that the assertion above is false. In that case, the edges around every vertex have both the values 1 and -1. From symmetry, we see that when we travel cyclically around each vertex, the number of sign changes of the edges encountered is ≥ 4. If s denotes the number of sign changes over all vertices, then $s \geq 4V$. On the other hand, we may count the sign changes as we travel along faces. As the number of sign changes

on a d-sided face is at the most d, we get

$$s \leq 2F_3 + 4F_4 + 4F_5 + 6F_6 + 6F_7 + \cdots$$

where F_d is the number of faces with d sides. As we clearly have

$$2E = 3F_3 + 4F_4 + \cdots,$$

Euler's formula gives

$$\begin{aligned} 4V - 8 = 4E - 4F &= 2(3F_3 + 4F_4 + \cdots) - 4(F_3 + F_4 + \cdots) \\ &= 2F_3 + 4F_4 + 6F_5 + \cdots \geq s \geq 4V \end{aligned}$$

a contradiction. Hence, the assertion holds.

MS 91 (1-2) 2022 : Problem 1 (Proposed by Demiri and Rexhepi). Prove that

$$\int_0^1 \left(t^{\frac{-1}{n}} - t^{1-\frac{1}{n}}\right)^{n-1} dt = \frac{n^n}{(n+1)(n+\frac{1}{2})(n+\frac{1}{3})\cdots(n+\frac{1}{n-1})}$$

where $n \in \mathbb{N}$ and $n > 1$.

Andrés Ventas, Santiago de Compostela, Galicia, Spain gave the solution to the problem which is presented below. **Dr. Henry Ricardo** Westchester Area Math Circle, New York, USA also gave a solution to the problem.

Solution

$$\int_0^1 \left(t^{\frac{-1}{n}} - t^{1-\frac{1}{n}}\right)^{n-1} dt = \int_0^1 \left(t^{\frac{-1}{n}}(1-t)\right)^{n-1} dt = \int_0^1 t^{\frac{1}{n}-1}(1-t)^{n-1}\, dt.$$

This is the beta function with $z_1 = \frac{1}{n}, z_2 = n$.

$$\begin{aligned} \mathcal{B}\Big(\frac{1}{n}, n\Big) = \frac{\Gamma(\frac{1}{n})\Gamma(n)}{\Gamma(\frac{1}{n}+n)} &= \frac{(-1+\frac{1}{n})!(n-1)!}{(\frac{1}{n}+n-1)!} \\ &= \frac{(-1+\frac{1}{n})!(n-1)!}{(n-1+\frac{1}{n})(n-2+\frac{1}{n})\cdots(1+\frac{1}{n})(\frac{1}{n})(-1+\frac{1}{n})!} \\ &= \frac{n^n(n-1)!}{(n(n-1)+1)(n(n-2)+1)\cdots(n+1)(1)} \\ &= \frac{n^n}{(n+\frac{1}{n-1})(n+\frac{1}{n-2})\cdots(n+1)}. \end{aligned}$$

We have got the required result.

MS 91 (1-2) 2022 : Problem 4 (Posed by Prof. Sury).
Let $f(x) = \cot(x)$. Note that $f'(x) = -f(x)^2 - 1$. More generally, for each $n \geq 0$, write

$$n!f(x)^{n+1} = a_n + b_{n0}f(x) + b_{n1}f'(x) + \cdots + b_{nn}f^{(n)}(x)$$

for some a_n's and b_{nm}'s.Find a recursion for the b_{ij}'s and determine all the a_n's.

The derivatives of $\cot(x)$ are

$$\cot(x),\ -\csc^2(x),\ 2\cot(x)\csc^2(x),\ -2\csc^4(x) - 4\cot^2(x)\csc^2(x),$$
$$16\cot(x)\csc^4(x) + 8\cot(x)^3\csc^2(x), \ldots$$

Substituting $\cot(x) = x, \csc^2(x) = \dfrac{1}{\sin^2(x)} = \dfrac{\sin^2(x) + \cos^2(x)}{\sin^2(x)} = 1 + x^2$, we have

$$x, -x^2 - 1, 2x^3 + 2x, -6x^4 - 8x^2 - 2, 24x^5 + 40x^3 + 16x, \ldots$$

We must solve n linear systems when we substitute into the statement's equation.

$x = a_0 + b_{00}x.$

$x^2 = a_1 + b_{10}x + b_{11}(-x^2 - 1).$

$2x^3 = a_2 + b_{20}x + b_{21}(-x^2 - 1) + b_{22}(2x^3 + 2x).$

$6x^4 = a_3 + b_{30}x + b_{31}(-x^2 - 1) + b_{32}(2x^3 + 2x) + b_{33}(-6x^4 - 8x^2 - 2).$

$24x^4 = a_4 + b_{40}x + b_{41}(-x^2 - 1) + b_{42}(2x^3 + 2x) + b_{43}(-6x^4 - 8x^2 - 2)$
$\quad + b_{44}(24x^5 + 40x^3 + 16x).$

$\ldots$

$$\begin{array}{c}
b_{00} = 1 \\
b_{10} = 0 \quad b_{11} = -1 \\
b_{20} = -2 \quad b_{21} = 0 \quad b_{22} = 1 \\
b_{30} = 0 \quad b_{31} = 8 \quad b_{32} = 0 \quad b_{33} = -1 \\
b_{40} = 24 \quad b_{41} = 0 \quad b_{42} = -20 \quad b_{43} = 0 \quad b_{44} = 1 \\
b_{50} = 0 \quad b_{51} = -184 \quad b_{52} = 0 \quad b_{53} = 40 \quad b_{54} = 0 \quad b_{55} = -1 \\
\ldots \quad \ldots
\end{array}$$

The coefficients for the b_n's appear in the unsigned triangle proposed by Wolfdieter Lang in [3] (A111594, Triangle of arctanh numbers), obtained as $T(n,k) =$ coefficients of x^n of $((\operatorname{arctanh}(x))^k)/k!$.

In our case with one row less and considering the signs, the recurrence is $T(n,k) = -T(n-1,k-1) - (n)(n-1)T(n-2,k),\ T(0,0) = 1$.

For the a_n's, we have the sequence: $0, -1, 0, 6, 0, -120, 0, 5040, \cdots$;

$a_n = -n(n-1)a_{n-2}$, with $a_0 = 0,\ a_1 = -1$. So, $a_n = n!$ for n odd and sign alternation.

References

[1] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

MS 91 (1-2) 2022 : Problem 5 (Posed by Prof. Sury).
Let a_n denote the number of different ways of putting brackets on a sequence of n objects. For example,

$$(p_1p_2p_3), (p_1)(p_2p_3), (p_1p_2)(p_3)$$

shows that $a_3 = 3$. Check for instance that $a_4 = 11$. Consider also the number b_n of paths from $(0,0)$ to (n,n) which never go above the line $y = x$ and where each step is a unit north, east or north-east (that is, cross). For instance, if we write h, v, c respectively for an eastward horizontal step, a northward vertical step and a southwest-northeast cross step, then

$$hvc, hcv, hhvv, hvhv, cc, chv$$

are the six paths from $(0,0)$ to $(2,2)$ which shows $b_2 = 6$. Prove that $b_n = 2a_{n+1}$ for all n.

Solution by Prof. Sury.
The numbers a_n are known as the small Schröder numbers. The recursive definition of putting brackets shows that the generating function $f(x) = \sum_n a_n x^n$ satisfies

$$f(x) = x + f(x)^2 + f(x)^3 + f(x)^4 + \cdots = x + \frac{f(x)^2}{1 - f(x)}.$$

Thus, $f(x)$ is a solution of the quadratic equation

$$2f(x)^2 - (1+x)f(x) + x = 0.$$

Solving for this and using the initial few values, we obtain

$$f(x) = \sum_{n\geq 1} a_n x^n = \frac{1}{4}(1 + x - \sqrt{x^2 - 6x + 1}).$$

The numbers b_n known as the larger Schröder numbers count paths. Looking at the number of paths that start with a diagonal step and those that start with a horizontal step, it is not difficult to show that b_n's satisfy the recursion

$$b_n = b_{n-1} + \sum_{k=0}^{n-1} b_k b_{n-1-k}.$$

Using this, the generating function $\sum_n b_n x^n$ can be obtained to be

$$\sum_{n\geq 0} b_n x^n = \frac{1}{2x}(1 - x - \sqrt{x^2 - 6x + 1}).$$

A comparison of the generating functions of a_n's and b_n's gives us the assertion $b_n = 2a_{n+1}$ for all $n \geq 0$.

MS 91 (1-2) 2022 : Problem 7 (posed by Prof. Sury).
Let $C(x)$ be the Cantor singular function; this is the unique non-decreasing function on $[0;1]$ such that if $x = \sum_{n\geq 1} a_n/3^n$ where a_n's are 0 or 2, then $C(x) = \sum_{n\geq 1} \frac{a_n/2}{2^n}$. Find the values of $\int_0^1 C(x)^n dx$ for $1 \leq n \leq 5$.

Andrés Ventas also gave the following solution to the problem.

Solution.

I have found two existing solutions to this problem, one from Gorin and Kukushkin from 2004 [2], and the other from Gordon from 2009 [1].

I will reproduce here a summary of Gordon's solution.

The intervals for the Cantor function with constant values are
$I_n = \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{9}, \frac{2}{9}\right), \left(\frac{7}{9}, \frac{8}{9}\right), \left(\frac{1}{27}, \frac{2}{27}\right), \left(\frac{7}{27}, \frac{8}{27}\right), \left(\frac{19}{27}, \frac{20}{27}\right), \left(\frac{25}{27}, \frac{26}{27}\right), \cdots$
The values of $C(x)$ in these intervals I_i are $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \cdots$.
Using the fact that the Cantor set has measure zero we can write

$$\int_0^1 C(x)^n dx = \sum_{i=1}^{\infty} \int_{I_i} C(x)^n dx$$

We rearrange the intervals and integrate easy constant functions.

$$\int_{I_i} C(x)^n dx = \frac{1}{3}\Big(\frac{1}{2}\Big)^n + \frac{1}{9}\Big(\Big(\frac{1}{4}\Big)^n + \Big(\frac{3}{4}\Big)^n\Big) + \frac{1}{27}\Big(\Big(\frac{1}{8}\Big)^n + \Big(\frac{3}{8}\Big)^n + \Big(\frac{5}{8}\Big)^n$$
$$+ \Big(\frac{7}{8}\Big)^n\Big) + \cdots$$
$$= \sum_{i=1}^{\infty}\left(\frac{1}{3^i}\sum_{j=1}^{2^{i-1}}\Big(\frac{2j-1}{2^i}\Big)^n\right) = \sum_{i=1}^{\infty}\left(\frac{1}{3^i}\frac{1}{2^{in}}\left(\sum_{j=1}^{2^i} j^n - \sum_{j=1}^{2^{i-1}}(2j)^n\right)\right) \tag{1}$$

The inner sum is a sum of odd powers of integers, thus we can write it as powers of all integers minus powers of even integers.

Now, we have the sum of powers of integers, for which Bernoulli obtained a formula in the form of polynomials (see Bernoulli numbers). In Gordon's paper they are written in a slightly unusual form for ease of notation.

$$\sum_{k=1}^{i} k^n = a_{n+1}i^{n+1} + a_n i^n + \cdots a_1 i = \sum_{j=1}^{n+1} a_j i^j = \sum_{j=-1}^{n-1} a_{n-j} i^{n-j},$$

where the coefficients a_j of the polynomial depend on n. We use it in 1 where we have two sums of powers of integers

$$\begin{aligned}\sum_{j=1}^{2^i} j^n - \sum_{j=1}^{2^{i-1}}(2j)^n &= \sum_{j=-1}^{n-1} a_{n-j}(2^{i(n-j)} - 2^n 2^{(i-1)(n-j)}) \\ &= \sum_{j=-1}^{n-1} a_{n-j}(2^{in}2^{-ij} - 2^n 2^{in} 2^{-ij} 2^{-n} 2^j) \\ &= 2^{in}\sum_{j=-1}^{n-1} a_{n-j} 2^{-ij}(1-2^j)\end{aligned}$$

And as a final step, we replace this result in 1 and apply the geometric sum

$$\begin{aligned}\int_0^1 C(x)^n dx &= \sum_{i=1}^{\infty}\left(\frac{1}{3^i}\frac{1}{2^{in}}\left(\sum_{j=1}^{2^i} j^n - \sum_{j=1}^{2^{i-1}}(2j)^n\right)\right) \\ &= \sum_{i=1}^{\infty}\left(\frac{1}{3^i}\sum_{j=-1}^{n-1}\frac{1}{2^{ij}}a_{n-j}(1-2^j)\right) = \sum_{j=-1}^{n-1} a_{n-j}(1-2^j)\sum_{i=1}^{\infty}\frac{1}{3^i\cdot 2^{ij}} \\ &= \sum_{j=-1}^{n-1}\frac{1-2^i}{3\cdot 2^j - 1}a_{n-j}\end{aligned}$$

Gordon represents the result in a nice matrix multiplication expression, for $0 \le n \le 5$,

$$\begin{bmatrix} \int_0^1 C^0 \\ \int_0^1 C^1 \\ \int_0^1 C^2 \\ \int_0^1 C^3 \\ \int_0^1 C^4 \\ \int_0^1 C^5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{5} & \frac{1}{2} & \frac{1}{3} & 0 & -\frac{1}{30} \\ \frac{1}{6} & \frac{1}{2} & \frac{5}{12} & 0 & -\frac{1}{12} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{5} \\ -\frac{3}{11} \\ -\frac{7}{23} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{3}{10} \\ \frac{1}{5} \\ \frac{33}{230} \\ \frac{5}{46} \end{bmatrix}$$

For the coefficients a_{n-j} of the polynomials for the sums of powers of integers, and Bernoulli numbers you can see the on-line articles [5] and [4].

References

[1] Russell A. Gordon, Some Integrals Involving the Cantor Function, The American Mathematical Monthly, (2009) **116:3**, 218–227,

[2] E. A. Gorin and B. N. Kukushkin, Integrals related to the Cantor function, St. Petersburg Math. J. **Vol. 15** (2004), No. 3 pages 449–468.

[3] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, https://oeis.org.

[4] Bernoulli number https://en.wikipedia.org/wiki/Bernoulli_number

[5] Faulhaber's formula https://en.wikipedia.org/wiki/Faulhaber%27s_formula

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MS 91 (1-2) 2022 : Problem 8 (Proposed by Prof. Sury).

For a positive real number x, let $AT(x)$ denote the value of $arctan(x)$ (that is, $tan^{-1}(x)$) in $[0, \pi/2]$. Observe $AT(3) = 3AT(1) - AT(2)$, $AT(8) = 2AT(1) + AT(3) - AT(5)$. Prove that a positive integer n has the property that $AT(n) = \sum_{i=1}^{n-1} a_i AT(i)$ for some integers a_i if, and only if, $n^2 + 1$ has a prime factor $p \ge 2n$.

Solution by Prof. Sury.

We show first that a positive integer n has the reducibility property that $AT(n) = \sum_{i=1}^{n-1} a_i AT(i)$ for some integers a_i if, and only if, all the prime factors of $1 + n^2$ occur among the prime factors of $1 + d^2$ as d varies in $1, \cdots, n-1$.

To prove this, we note that it can be shown that the equality $AT(n) =$

$\sum_r AT(n_r)$ where n_r are not necessarily distinct, is equivalent to

$$arg(1+in) = \prod_r arg(1 \pm in_r).$$

In other words, $\frac{\prod_r (1-in_r}{1+in}$ is rational. Writing the numerator as $u+iv$, we have by taking absolute values that $u^2(1+n^2) = \prod_r (1+n_r^2)$. This shows the necessity of the asserted condition. The sufficiency follows by observing that if every prime factor of $1+n^2$ as a prime dividing $1+d^2$ for $1 \leq d < n$, we may write $1+in$ as a product in $\mathbb{Z}[i]$ and obtain a relation of the form $(1+in)\prod_k (u_k^2+v_k^2) = \prod_r (1+in_r)^{a_r}$ - this requires some calculation which is involved but routine. Comparing arguments, we get the sufficiency.
Finally, to deduce our original assertion from the one we proved now, suppose every prime factor of $1+n^2$ is $< 2n$. Then, for every prime p dividing $1+n^2$, writing s_p for the smallest number such that $1+s_p^2$ is 0 mod p, we have $s_p \leq (2n-1)/2 < n$. This proves that each prime p dividing $1+n^2$ divides $1+d^2$ for some $d < n$. Thus, by the assertion n is a number admitting a reduction as in the problem. Finally, assume a decomposition for n as in the problem. Then the assertion we proved shows that if a prime p dividing $1+n^2$ is $\geq 2n$, then it divides $1+n_p^2$ and $n_p < n$. So $n^2 - n_p^2 = (n-n_p)(n+n_p)$ is a multiple of p. But this is a contradiction as one factor is less than n and the other less than $2n$. Thus, the problem follows from the assertion which we proved.

MS 91 (1-2) 2022 : Problem 9 (Proposed by Dr. Anup Dixit).
Let $f : \mathbb{N} \to \mathbb{N}$ be a function on positive integers satisfying the property

$$f(f(n)) + f(n+1) = n+2$$

Show that $f(n) = \lfloor n\alpha \rfloor + 1$, where $\alpha = \dfrac{\sqrt{5}-1}{2}$.
Solution by Andres ventas.
Let $\lfloor x \rfloor$ and $\{x\}$ denote the integer and fractional parts of a number x, so $x = \lfloor x \rfloor + \{x\}$ and also $\lfloor x \rfloor = x - \{x\}$.

It is also easy to verify the following identities

$$\alpha^2 = 1-\alpha; \quad \alpha^2+\alpha = 1; \quad \alpha^2+\alpha^3 = \alpha; \tag{2}$$

We substitute the definition of $f(n)$ into the equality of the statement

$$
\begin{aligned}
& f(f(n)) + f(n+1) = n+2; \\
& \lfloor (\lfloor n\alpha \rfloor + 1)\alpha \rfloor + 1 + \lfloor (n+1)\alpha \rfloor + 1 = n+2; \\
& \lfloor \lfloor n\alpha \rfloor \alpha + \alpha \rfloor + \lfloor n\alpha + \alpha \rfloor = n; \\
& \lfloor n\alpha^2 - \{n\alpha\}\alpha + \alpha \rfloor + \lfloor \lfloor n\alpha \rfloor + \{n\alpha\} + \alpha \rfloor = n; \\
& \lfloor n(1-\alpha) - \{n\alpha\}\alpha + \alpha \rfloor + \lfloor \lfloor n\alpha \rfloor + \{n\alpha\} + \alpha \rfloor = n; \\
& n + \lfloor -\lfloor n\alpha \rfloor - \{n\alpha\} - \{n\alpha\}\alpha + \alpha \rfloor + \lfloor n\alpha \rfloor + \lfloor \{n\alpha\} + \alpha \rfloor = n; \\
& -\lfloor n\alpha \rfloor + \lfloor -\{n\alpha\} - \{n\alpha\}\alpha + \alpha \rfloor + \lfloor n\alpha \rfloor + \lfloor \{n\alpha\} + \alpha \rfloor = 0;
\end{aligned}
$$

$$\lfloor -\{n\alpha\} - \{n\alpha\}\alpha + \alpha \rfloor + \lfloor \{n\alpha\} + \alpha \rfloor = 0; \tag{3}$$

Now, we calculate the value of both floor values which are dependent on the value of the fractional part $\{n\alpha\}$.

For the second floor value $\lfloor v_2 \rfloor = \lfloor \{n\alpha\} + \alpha \rfloor$, we use 2: $\alpha + \alpha^2 = 1$.

For $\{n\alpha\} > \alpha^2$; $v_2 > \alpha^2 + \alpha = 1$; and $v_2 < 1 + \alpha < 2$ implies $\lfloor v_2 \rfloor = 1$.

For $\{n\alpha\} < \alpha^2$; $v_2 < \alpha^2 + \alpha = 1$; and $v_2 > 0 + \alpha > 0$ implies $\lfloor v_2 \rfloor = 0$.

$\{n\alpha\}$ can not be equal to α^2 because we would have $(n+1)\alpha \in \mathbb{N}$, which is impossible.

For the first floor value $\lfloor v_1 \rfloor = \lfloor -\{n\alpha\} - \{n\alpha\}\alpha + \alpha \rfloor$, we use 2: $\alpha + \alpha^2 = 1$ and $\alpha^2 + \alpha^3 = \alpha$.

For $\{n\alpha\} > \alpha^2$; $v_1 < -\alpha^2 - \alpha^3 + \alpha = 0$; and $v_1 > -1 - \alpha + \alpha > -1$ implies $\lfloor v_1 \rfloor = -1$.

For $\{n\alpha\} < \alpha^2$; $v_1 > -\alpha^2 - \alpha^3 + \alpha = 0$; and $v_1 < 0 + \alpha < 1$ implies $\lfloor v_1 \rfloor = 0$.

In both cases we have $v_1 + v_2 = 0$, equation 3 holds, and thus the equality in the statement holds, completing the proof.

MS 91 (1-2) 2022: Problem 12 (Proposed by Dr. Siddhi Pathak).
Let

$$I = \int_0^\infty \frac{(\log x)^4}{1+x^2}\, dx.$$

Prove that I is an algebraic number times π^5.

Dr. Henry Ricardo, Westchester Area Math Circle, New York, USA and Mr. **Andrés Ventas** provided correct solutions to the problem. The solution provided by Dr. Henry Ricardo is presented below.

Solution by Dr. Henry Ricardo.

We show that $I = (5/32)\pi^5$.

First we recall Euler's *reflection formula*

$$\Gamma(s)\Gamma(1-s) = \int_0^\infty \frac{y^{s-1}}{1+y}\, dy = \frac{\pi}{\sin(\pi s)},\ s \notin \mathbb{Z}.$$

Using Leibniz's formula for differentiating under the integral sign, we have

$$\frac{d^4}{ds^4}\left(\frac{\pi}{\sin(\pi s)}\right) = \frac{d^4}{ds^4}\int_0^\infty \frac{y^{s-1}}{1+y}\, dy = \int_0^\infty \frac{\partial^4}{\partial s^4}\left(\frac{y^{s-1}}{1+y}\right) dy,$$

or

$$\frac{\left(\cos^4(\pi s) + 18\cos^2(\pi s) + 5\right)}{\sin^5(\pi s)} \cdot \pi^5 = \int_0^\infty \frac{(\log y)^4\, y^{s-1}}{1+y}\, dy. \tag{4}$$

Taking the limit of both sides of (2) as $s \to 1/2$, we have

$$5\pi^5 = \int_0^\infty \frac{(\log y)^4}{\sqrt{y}\,(1+y)}\, dy \overset{\sqrt{y}=x}{=} 2^5 \int_0^\infty \frac{(\log x)^4}{1+x^2}\, dx.$$

Dividing by 32 completes the proof.

MS 91 (3-4) 2023 : Problem 1 (Proposed by Dr. Anup Dixiy)
Suppose a_n is a sequence of positive integers such that $\sum_{n=1}^{\infty} \dfrac{\sin(1/a_n)}{\log a_n}$ diverges. Show that for infinitely many n, $lcm\{a_1 \cdots a_n\} = lcm\{a_1 \cdots a_{n+1}\}$.

Solution by Andrés Ventas.

If we want to get a sequence of integers a_n that do not meet the equality of their lcm, we need each new number a_{n+1} in the sequence to be a prime or a power of prime. If we are forced to include other numbers in that sequence, those numbers will have common factors with the primes and their powers, and we will have the equality of their lcm.

The statement says that the series diverges so we must prove that the series of primes and their powers converges because that would force us to include other integers to achieve divergence. Moreover, this new subset of integers should be infinite because a finite sum would be convergent.

In short, to obtain a proof of the problem we must show that the sum $\sum_{n=1}^{\infty} \dfrac{\sin(1/a_n)}{\log a_n}$ with the positive integers $\{a_1 \cdots a_n\}$ that are prime, and their powers, is convergent.

First we prove the convergence for the prime numbers. We have that $\sin(1/x) < 1/x$, (for low numbers $\sin(1/x) \approx 1/x$). We also have $p_n >$

$n \log n$, (with an asymptotic approximation $n \log n \sim p_n$, by the prime number theorem).

We replace both inequalities and we use the Cauchy condensation test. It can be used for a non-increasing sequence $f(n)$ of non-negative real numbers, as is the case with our sequence $1/(a_n \log a_n)$.

$$\sum_{n=1}^{\infty} \frac{\sin(1/p_n)}{\log p_n} < \sum_{n=1}^{\infty} \frac{1}{p_n \log p_n} < \sum_{n=1}^{\infty} \frac{1}{n(\log n)^2}$$

Using Cauchy condensation test,

$$\sum_{n=1}^{\infty} \frac{1}{n(\log n)^2} < \sum_{n=1}^{\infty} 2^n \frac{1}{2^n (\log 2^n)^2} = \sum_{n=1}^{\infty} \frac{1}{(\log 2^n)^2} = \frac{1}{(\log 2)^2} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and the series converges.

And now we prove the convergence for the powers of the primes

$$\sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{p_n^k \log p_n^k} = \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} \frac{1}{k p_n^k \log p_n} < \sum_{n=1}^{\infty} \frac{1}{\log p_n} \sum_{k=2}^{\infty} \frac{1}{p_n^k}$$
$$= \sum_{n=1}^{\infty} \frac{1}{\log p_n} \frac{1}{p_n^2} \frac{1}{1 - \dfrac{1}{p_n}} = \sum_{n=1}^{\infty} \frac{1}{(\log p_n)(p_n^2 - p_n)} < \sum_{n=1}^{\infty} \frac{1}{p_n \log p_n},$$

and the series converges because it is less than the series of primes.

So, the sum of both series also converges and the proof is completed.

MS 91 (3-4) 2022 : Problem 2 (Proposed by Dr. Anup Dixit).

Let $a_1, a_2, \cdots, a_n$ and $a_1, a_2, \cdots, a_n$ be two permutations of $1, 2, \cdots, n$. Show that the set $a_1 b_1, a_2 b_2, \cdots, a_n b_n$ does not form a complete residue modulo n.

Andrés Ventas gave a solution to this problem. However, the solution provided by **Dr. Anup Dixit** is shorter, elegant and is presented below.

Solution by Dr. Anup Dixit.

Suppose $\{a_1 b_1, a_2 b_2, \cdots, a_n b_n\}$ forms a complete residue system modulo n. Then, for any fixed index i, we first show that if $d \mid a_i$, then $d \mid b_i$ for any $1 \le d \le n$. Indeed, the number of multiples of d in $\{1, 2, \cdots, n\}$ is $\lfloor \frac{n}{d} \rfloor$. If $d \mid a_i$ and $d \nmid b_i$, there are $\lfloor \frac{n}{d} \rfloor$ indices j, with $j \ne i$, for which $d \mid b_j$. Thus, we obtain greater than $\lfloor \frac{n}{d} \rfloor$ multiples of d in the set $\{a_1 b_1, \cdots, a_n b_n\}$, which is a contradiction.

Since $d \mid a_i \implies d \mid b_i$ for all $1 \leq d \leq n$, we conclude that $a_i = b_i$. Thus the set $\{a_i b_i\}$ consists only of the quadratic residues modulo n, which is not a complete residue system modulo n.

MS 91 (3-4) 2022 : Problem 3 (Proposed by Dr. Mohsen Soltanifar) Let X be a real valued random variable on the real line with finite mean. Assume for some $-\infty < \alpha < \infty$ we have:

$$E(min(X, \alpha)) = E(max(X, \alpha)).$$

Calculate the distribution of X.

Solution by Andrés Ventas.

Since $E(.)$ is order preserving, from $max(X, \alpha) \geq X$ and $max(X, \alpha) \geq \alpha$ it follows $E(max(X, \alpha)) \geq E(X)$ and $E(max(X, \alpha)) \geq E(\alpha) = \alpha$.

With both results we have $E(max(X, \alpha)) \geq max(E(X), \alpha)$. In the same way for the minimum we have $E(min(X, \alpha)) \leq min(E(X), \alpha)$.

Replacing both inequalities in the statement of the problem, we have $min(E(X), \alpha) = max(E(X), \alpha)$, thus $E(X) = \alpha$. Therefore,

$$E(min(X, E(X)) = E(max(X, E(X)) \tag{5}$$

$$X = E(x) = \alpha. \tag{6}$$

The distribution of X is a degenerate distribution that takes only a single value α; $P(X = \alpha) = 1$.

We are going to prove 6 by contradiction:

if $X < E(X)$ then $X = min(X, E(X)) < max(X, E(X)) = E(X)$ and $E(min(X, E(X))) < E(max(X, E(X)))$ in contradiction with 5;

if $X > E(X)$ then $E(X) = min(X, E(X)) < max(X, E(X)) = X$ and $E(min(X, E(X))) < E(max(X, E(X)))$ also in contradiction with 5.

MS 91 (3-4) 2022 : Problem 5 (Proposed by Yathiraj Sharma).

Consider the sequence $d_n = 3n+1$. Prove that the sum of the Legendre symbols $(k/7)$ as k runs through divisors of $12(7d-4)$ is 0 whenever $d \neq d_n$. Show further that for infinitely many (but not all) n, the sum is not 0 as k runs over divisors of $12(7d_n - 4)$.

Solution by Andrés Ventas.

The quadratic residues of $p = 7$ are $\{1, 2, 4\}$.

The divisors of 12 are $d_{12} = \{1, 2, 3, 4, 6, 12\}$, so the Legendre symbols of d_{12} are $\left(\dfrac{d_{12}}{7}\right) = \{1, 1, -1, 1, -1, -1\}$ because the Legendre symbols for the prime p have a cycle equal to p, therefore $\left(\dfrac{12}{7}\right) = \left(\dfrac{5}{7}\right) = -1$.

We see that the sum of the elements of $d_{12} = 0$, and we know that the Legendre symbol is a completely multiplicative function of its top argument: $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

We also see that $(7(3n + 1) - 4)$ can be written as $(7 \cdot 3n + 3) = 3(7n + 1)$, so the factor $f_{dn} = 7d_n - 4$ has a factor $3^r, r \geq 1$, and the factor $f_d = 7d - 4, d \neq d_n$ has no factor 3.

We obtain the prime factorization of the factor $f_d = p_1^{a_1} \cdots p_n^{a_n}$, and we obtain a new subset of divisors of $m = 12 \cdot f_d$ by multiplying the divisors of 12 by an odd prime power $p_i^{a_i}, a_i \geq 1,$. Thus, we obtain a subset of Legendre symbols $\{1, 1, -1, 1, -1, -1\}$ or $\{-1, -1, 1, -1, 1, 1\}$, both with sum zero. This is because the prime number has the Legendre symbol 1 or -1 (it is coprime with 7), and by the multiplicative property, we have the same symbols as 12 or the same symbols with sign alternation.

If the factor is a power of 2, or a product with a power of 2, then we have $\{2, 4\}$ as common factors with d_{12}; therefore, we have some common divisors: $2 \cdot d_{12} = \{2, 4, 6, 8, 12, 24\}$ and $4 \cdot d_{12} = \{4, 8, 12, 16, 24, 48\}$. Here the only new divisors of $2 \cdot d_{12}$ are $\{8, 24\}$, and for $4 \cdot d_{12}$, they are $\{16, 48\}$, both multiples of $\{4, 12\}$. The Legendre symbols of $\{4, 12\}$ are $\{1, -1\}$ with sum zero, and due to the multiplicative property, the subsets of divisors that we get multiplying by a power of 2 also have sum zero, in the same way that we saw for the odd primes.

For the second part of the statement we have the factor $f_{dn} = 7d_n - 4 = 3(7n + 1)$. We have the sequence $7n + 1 = 1, 8, 15, 22, 29, 36, 43, \ldots$. Thus, we have an additional 3 factor for $i = 3n - 1$, and we have no 3 factor for the others n.

With no additional 3 we have one 3 in d_{12} and other in f_{dn}, so $3 \cdot d_{12} = \{3, 6, 9, 12, 18, 36\}$ and the not commons divisors are $\{9, 18, 36\}$ with the Legendre symbols $\{1, 1, 1\}$. Thus, with odd powers of 3 in f_{dn} and only other prime as a factor, we have infinity n with no sum zero.

With an additional 3 we have $9 \cdot d_{12} = \{9, 18, 27, 36, 54, 108\}$ and the not commons divisors are $\{27, 54, 108\}$ with the Legendre symbols $\{-1, -1, -1\}$.

In these cases, we have that the three new divisors that we get with 3 are compensated by the three new divisors that we get with 9. Thus, with even powers of 3 in f_{dn}, we also have infinity n with sum zero.

MS 91 (3-4) 2022 : Problem 6 (Posed by Prof. Rexhepi and Ilir Demiri).

Prove that

$$\int_0^\infty \frac{u^3 du}{e^{\sqrt[4]{\frac{15}{4}}u} - 1} = \frac{4\pi^4}{225}$$

Dr. Henry Ricardo and Andrés Ventas gave solutions to this problem. We present below the solution provided by Dr. Ricardo.

Solution by Dr. Henry Ricardo.

With the substitution $t = \sqrt[4]{15/4}\ u$, the given integral becomes

$$\frac{4}{15}\int_0^\infty \frac{t^3}{e^t - 1}\,dt.$$

Expanding the integrand, we have

$$\frac{t^3}{e^t-1} = t^3 \cdot \frac{e^{-t}}{1-e^{-t}} = \sum_{k=0}^{\infty} t^3 e^{-(k+1)t},$$

so that

$$\frac{4}{15}\int_0^\infty \frac{t^3}{e^t-1}\,dt = \frac{4}{15}\int_0^\infty \sum_{k=0}^{\infty} t^3 e^{-(k+1)t}\,dt = \frac{4}{15}\sum_{k=0}^{\infty}\int_0^\infty t^3 e^{-(k+1)t}\,dt,$$

where the interchange of integration and summation is justified by the monotone convergence theorem. Now the change of variable $v = (k+1)t$ transforms the last expression into

$$\begin{aligned}
\frac{4}{15}\sum_{k=0}^{\infty}\int_0^\infty \left(\frac{v}{k+1}\right)^3 e^{-v}\cdot\frac{1}{k+1}\,dv \;&=\; \frac{4}{15}\sum_{k=0}^{\infty}\frac{1}{(k+1)^4}\cdot\int_0^\infty v^3\,e^{-v}\,dv \\
&=\; \frac{4}{15}\cdot\zeta(4)\cdot\Gamma(4) \\
&=\; \frac{4}{15}\cdot\frac{\pi^4}{90}\cdot 6 \;=\; \frac{4\pi^4}{225},
\end{aligned}$$

where ζ denotes the Riemann zeta function, Γ denotes the gamma function, and we have used known values of these functions.

MS 91 (3-4) 2022 : Problem 7 (posed by Prof. Rexhepi and Ilir Demiri)
For $a > b > e$, e-Euler number, probe that

$$\frac{\ln \Gamma(b^a)}{\ln \Gamma(b^a)} > \frac{\ln b}{\ln a}.$$

Shubhayan Ghosal, Jadavpur Univeristy, Kolkata and **Andrés Ventas** have provided almost the same solution to the problem. The solution is given below.

Solution.

The derivative of the function $f(x) = \frac{\ln x}{x}$ is $f'(x) = \frac{1-\ln x}{x^2}$. Thus, $f(x) = \frac{\ln x}{x}$ is decreasing for $x \geq e$. It implies that, for $a > b > e$,

$$\frac{\ln b}{b} > \frac{\ln a}{a}; \quad a \ln b > b \ln a; \quad b^a > a^b.$$

Since the $\Gamma(x)$ and $\ln(x)$ functions are increasing, it leads to

$$\ln \Gamma(b^a) > \ln \Gamma(a^b).$$

With that result the final step is

$$\frac{\ln \Gamma(b^a)}{\ln \Gamma(a^b)} > 1 \text{ and } \frac{\ln b}{\ln a} < 1,$$

which completes the proof.

MS 91 (3-4) 2022 : Problem 8 (Proposed by Toyesh P. Sharma).

If $n > 0$ and α is the positive root of quadratic equation $x^2 - x - 1 = 0$ then show that the following inequality

$$F_n \alpha^{F_n} + L_n \alpha^{L_n} \geq 2F_{n+1} \alpha^{F_{n+1}}$$

holds.

Further, obtain the above inequality using the convexity of a suitable function where the Fibonacci numbers F_n and the Lucas numbers L_n satisfy the usual recurrence relations.

Andrés Ventas gave a solution to this problem. The solution is presented below.

Solution.

$F_n : 0, 1, 1, 2, 3, 5, 8, 13 \ldots$
$L_n : 2, 1, 3, 4, 7, 11, 18, 29 \ldots$

For $n = 1$ we have the equality, for $n = 2, (13.8 > 10.4)$ and $n = 3, (32.6 > 25.4)$ it is easy a numerical proof.

For $n > 3$, we will use the identity $L_n + 2 \geq F_{n+1}$ then it is sufficient to prove that $L_n\alpha^{L_n} \geq 2F_{n+1}\alpha^{F_{n+1}}$ holds.

Proof. Taking logarithms

$$\ln L_n + L_n \ln\alpha \geq \ln 2 + \ln F_{n+1} + F_{n+1}\ln\alpha$$

$$\ln(F_{n+1}+2) + (F_{n+1}+2)\ln\alpha \geq \ln 2 + \ln F_{n+1} + F_{n+1}\ln\alpha$$

$$\ln\alpha^2 + \ln(F_{n+1}+2) + F_{n+1}\ln\alpha \geq \ln 2 + \ln F_{n+1} + F_{n+1}\ln\alpha$$

and the proof is complete because $\alpha^2 > 2$. □

To obtain the inequality with the convexity of a suitable function we use the Jensen's inequality : If $p_1, ..., p_n$ are positive numbers which sum to 1 and f is a real continuous function that is convex, then

$$\sum_{i=1}^{n} p_i f(x_i) \geq f\Big(\sum_{i=1}^{n} p_i x_i\Big)$$

We take the convex function $f(x) = x\alpha^{2x}$, and the values $p_1 = p_2 = \frac{1}{2}$, $x_1 = F_{n-1}$, and $x_2 = F_n$.

$$\frac{1}{2}F_{n-1}\alpha^{2F_{n-1}} + \frac{1}{2}F_n\alpha^{2F_n} \geq \frac{1}{2}(F_{n-1}+F_n)\alpha^{(F_{n-1}+F_n)}$$

$$2F_{n-1}\alpha^{2F_{n-1}} + 2F_n\alpha^{2F_n} \geq 2F_{n+1}\alpha^{(F_{n+1})}$$

Now, we operate on the LHS of the problem statement. It is easy to prove that $\alpha^2 > 1$, $\alpha F_{n+1} > 2F_n$. And also, that $L_n \geq 2F_n+1$ and $L_n > 2F_{n-1}+2$ when $n > 2$. Thus,

$$\begin{aligned} F_n\alpha^{F_n} + L_n\alpha^{L_n} &= F_n\alpha^{F_n} + (F_{n-1}+F_{n+1})\alpha^{L_n} \\ &> F_{n-1}\alpha^{2F_{n-1}+2} + F_{n+1}\alpha^{2F_n+1} \\ &= \alpha^2 F_{n-1}\alpha^{2F_{n-1}} + \alpha F_{n+1}\alpha^{2F_n} \\ &> 2F_{n-1}\alpha^{2F_{n-1}} + 2F_n\alpha^{2F_n}. \end{aligned}$$

Joining both results, and with the already calculated values for $n \leq 3$, we obtain the original inequality,

$$F_n\alpha^{F_n} + L_n\alpha^{L_n} \geq 2F_{n-1}\alpha^{2F_{n-1}} + 2F_n\alpha^{2F_n} \geq 2F_{n+1}\alpha^{(F_{n+1})},$$

implies that, $F_n\alpha^{F_n} + L_n\alpha^{L_n} \geq 2F_{n+1}\alpha^{(F_{n+1})}$.

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